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# SATURATED BOOLEAN ALGEBRAS WITH ULTRAFILTERS

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## Introduction

In this paper we consider two topics: saturated atomless Boolean algebras, and the first order theory of Boolean algebras together with $\lambda$ ultrafilters ($\lambda$ is a cardinal).

Concerning the first theme, we have found several equivalent conditions for an atomless Boolean algebra to be $k$-saturated. The first is Negrepontis' separation principle $H_k$ which appeared in his 1969 paper [12 b]. However, for $k=\omega_1$ this condition was considered by W. Rudin [14], and for arbitrary $k$ and ordered sets, several separation (or intercallation) principles of the same nature were introduced by Đ. Kurepa [10 b]. Kurepa considered these properties in connection with $k$-universal ordered sets, and Negrepontis introduced $H_k$ in order to describe universal homogeneous Boolean algebras. The second condition is $k$-injectivity, where the term injectivity is taken in a stronger sense than it is usually considered in the theory of modules, or in Boolean algebras (cf. [7], [14]), i. e. all mappings in question are assumed to be 1—1. In fact, this notion is closely related to $k$-objective, and injective mappings in [19]. However, we rather used Blum's criterion on model completions (cf. [14] p. 89) than the approach of Yasuhara. Other properties, especially concerning cardinality of saturated Boolean algebras, are found. For example, it is shown that every infinite homomorphic image of an $\omega_1$-saturated Boolean algebra has cardinality $\geqslant 2^{\omega}$.

It the second part, the main theorem is: The theory $T_\lambda$ of Boolean algebras with $\lambda$ distinct ultrafilters has a model completion $T_\lambda^*$. Models of $T_\lambda^*$ are again atomless Boolean algebras. $k$-saturated models of $T_\lambda^*$ are described, those are axactly models $(\underset{\sim}{A}, U_\alpha)_{\alpha<\lambda}$, where $\underset{\sim}{A}$ is a Boolean algebra which satisfies $H_k$, and $U_\alpha$ are $k$-directed filters, i. e. $P(k)$ points in the Stone space of $\underset{\sim}{A}$.

In applications, we proved that the theory of distributive lattices has a model-completion, that is the complete theory of $(U \cap I, \cap, \cup, \subseteq)$, where $U$

is an ultrafilter, and $I$ a maximal ideal of an atomless Boolean algebra. Using a theorem of S. Shelah (cf. [16)]), and $\omega_1$-saturation of $2^\omega/F$, it was proved that the filter $F$ of cofinite sets is $\omega_1$-saturative. Posssibly new proofs of the existence of independent families of sets is given. In topological interpretations in the light of Stone functor, it is not a surprise that some well known theorems are obtained, as the ones of W. Rudin [14], and I. I. Parovičenko [13], respectively.

Many results in this paper are known through the theory of the Jónsson class of Boolean algebras (cf. [4]), or the theory of $N^* = \beta N - N$ (cf. [18]), but we think that the contribution should be seen in the unified treatment of the mentioned topics by methods of model theory.

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## 1. Preliminary.

The terminology which is used in this paper is according to [3], [11], and [15]. Thus, a language is denoted by $L$, the language of a theory $T$ by $L(T)$, and of a model $\mathfrak{C}$ by $L(\mathfrak{C})$. Universes of models $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{G}$, ... are denoted respectively by $A, B, C, \ldots$, and the cardinal number of $A$ by $|A|$. The class of all models of a theory $T$ is denoted by $\mathfrak{M}(T)$, and elementary diagram of $\mathfrak{C}$ by $\Delta(\mathfrak{C})$. Basic model-theoretic notions, like model-complete, model--completion, saturated models etc., and also related facts (definitions, theorems) are assumed to be known.

Further, we suppose that a model $\mathfrak{A}$ of a theory $T$ is a prime (universal) model iff it is embedable into every model of $T$ (every model of $T$ of cardinality $\leqslant |A|$ is embedable into $\mathfrak{A}$). Similar definition applies for homogeneous models: Any partial isomorphism $f: \mathfrak{A} \simeq_p \mathfrak{A}$, $|f| < |A|$, with domf$\subseteq \mathfrak{A}$, can be extended to an automorphism of $\mathfrak{A}$. Models which are called in [3] universal, homogeneous etc., we call elementary universal, elementary homogeneous etc.

Boolean algebras are denoted by $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \ldots$, and their domains respectively by $A, B, C$. It is assumed that every Boolean algebra $\underset{\sim}{B}$ is of the form $\underset{\sim}{B} = (B, +, \cdot, ', \leqslant, 0, 1)$, where $+$, $\cdot$, and $'$ are usual Boolean operations. Two--elements Boolean algebra is denoted by $\underset{\sim}{2}$. Instead of Boolean algebra, we shall often write shortly $BA$.

Let $\underset{\sim}{B}$ be a $BA$ and assume $X, Y \subseteq B$. The infimum of the set $X$ is denoted by $\prod_{x\in X} x$, and supremum by $\sum_{x\in X} x$. If $a \in B$, then $a < X$ stands for $\forall y \in X (a < y)$, and $a \leqslant X$ for $\forall y \in X$ $(a \leqslant y)$. By $X < Y$ we mean $\forall x \in X \forall y \in Y$ $(x < y)$. The similar meaning has $X \leqslant Y$. If $x \in B$, then $a \| x$ stands for $\neg a \leqslant x \wedge \neg x \leqslant a$, and $a \| X$ for $\forall y \in X (a \| y)$. By $X |< Y$, the formula $\forall x \in X \forall y \in Y$ $(\neg x \leqslant y)$ is denoted.

Now we state two lemmas on extensions of Boolean embeddings. A form of the first lemma appears in [7] p. 141.

Definition 1.1. A model $\mathfrak{B}$ is a simple extension of a model $\mathfrak{A}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$, and there is an element $a \in B$ such that $\mathfrak{B}$ is generated by $A \cup \{a\}$. (cf. [15] p. 89).

If $\mathfrak{B}$ is a simple extension of $\mathfrak{A}$, we shall write $\mathfrak{B}=\mathfrak{A}(a)$. In the case of *BA's* we use the notation $\underset{\sim}{B}=\underset{\sim}{A}(a)$.

Lemma 1.2. *Let $f: \underset{\sim}{A} \rightarrow \underset{\sim}{B}$ be a Boolean homomorphism, and $\underset{\sim}{A}(a)$ a simple extension of $\underset{\sim}{A}$. Suppose $f$ satisfies the following conditions for an element $b$ of $B$:*

$$\forall x \in A (x \leqslant a \Rightarrow f(x) \leqslant b), \quad \forall x \in A (a \leqslant x \Rightarrow b \leqslant f(x)).$$

*Then there is a unique homomorphism $h: \underset{\sim}{A}(a) \rightarrow \underset{\sim}{B}$ so that $f \subseteq h$ and $h(a)=b$.*

*If in addition $f$ is an embedding which satisfies $\forall x \in A (x \| a \Rightarrow f(x) \| b)$, then $h$ is also an ambedding.*

Proof: Elements of $A(a)$ are of the form $ax+a'y$, $x, y \in A$. The map $h(ax+a'y)=bf(x)+b'f(y)$ is well defined, and satisfies the required conditions. $\dashv$

Lemma 1.3. *Let $\underset{\sim}{A}$ be a finite BA, $\underset{\sim}{B}$ an atomless BA, and $h: \underset{\sim}{A} \rightarrow \underset{\sim}{B}$ an embedding. Then $h$ can be extended to an embedding $f: \underset{\sim}{A}(a) \rightarrow \underset{\sim}{B}$.*

Proof: In the following, we may assume without loss of generality that $h$ is an inclusion, i. e. $h: \underset{\sim}{A} \subseteq \underset{\sim}{B}$. Let $\{a_1, \ldots, a_n\}$ be the set of all atoms of $A$. Since $B$ is atomless, for every $a_i$ there is $c_i \in B$ such that $a_i > c_i > 0$. Let $I=\{i \leqslant n: aa_i \neq 0,\ a' a_i \neq 0\}$, $J=\{i \leqslant n: aa_i \neq 0,\ a' a_i=0\}$, and $b=\sum_{i \in I} c_i+\sum_{j \in J} a_j$. The map $f: A(a) \rightarrow B$ defined by $f(ax+a'y)=bx+b'y$ extends $h$, and it is a Boolean embedding. $\dashv$

Now we give a short rewiew of some well known facts on atomless Boolean algebras. All these assertions easily follow from the last lemma. Here $T$ denotes the theory of BA's, and $T^*$ the theory of atomless BA's.

Theorem 1.4. *$T^*$ is $\omega$-categorical.* $\dashv$

Theorem 1.5. *Every two models of $T^*$ are elementary equivalent in $\mathfrak{L}_{\infty\omega}$. $T^*$ is a complete theory.* $\dashv$

Theorem 1.6. *$T^*$ is model-complete.* $\dashv$

Theorem 1.7. *$T^*$ is model-completion of $T$* (cf. [8] p. 136). $\dashv$

Theorem 1.8. *$T^*$ is a submodel-complete theory.* $\dashv$

Theorem 1.9. *$T^*$ allows elimination of quantifiers.* $\dashv$

## 2. Saturated atomless Boolean algebras.

We give several equivalent descriptions of saturated atomless BA's. It appears that these BA's are exactly universal homogeneous BA's. The later are investigated in details in [4], and we shall use occasionally the technique and results contained in that book. However, our approach is different, and consists of applications of properties of theories which have model-completions. This approach in general setting is investigated in [11].

By a Jónsson class we mean any class of models which satisfies the conditions in [4], p. 84.

Universal homogeneous models will be called shortly *full* models.

The following theorems were proved in [11].

Theorem 2.1. ([11], Cor. 2.3) Assume that a theory $\Gamma$ has a model-completion $\Gamma^*$, $\forall\exists$ axiomatization, and a prime model. Then the class of all models of $\Gamma$, $\mathfrak{M}(\Gamma)$, is a Jónsson class. $\dashv$

Theorem 2.2. (cf. [11], T. 2.8.). Assume a theory $\Gamma$ has a model-completion $\Gamma^*$, a prime model, and suppose $L(\Gamma)$ is countable. Then:

1° If $\mathfrak{C}$ is an infinite saturated model of $\Gamma^*$, then $\mathfrak{C}$ is a full model of $\Gamma$.

2° If $\mathfrak{C}$ is a full model of $\Gamma$ of cardinality $k \geqslant \omega_1$, then $\mathfrak{C}$ is a saturated model of $\Gamma^*$.

Henceforth $T$ denotes the theory of BA's, and $T^*$ the theory of atomless BA's.

Proposition 2.3. $\mathfrak{M}(T), \mathfrak{M}(T^*)$ *are Jónsson classes.* (cf. [4], § 6)

Proof: By T. 1.7. and T. 2.1. Obviously, $T^*$ is the model-completion of itself. $\dashv$

By T. 1.7. and T. 2.2. we have immediately the following:

Proposition 2.4. *A Boolean algebra* $\underset{\sim}{B}$ *is universal-homogeneous* (i. e. a full BA) *iff* $\underset{\sim}{B}$ *is atomless and saturated.*

Now we proceed to more explicit description of full Boolean algebras.

Let $k$ be an infinite cardinal and $\underset{\sim}{A}$ a $BA$. The following useful conditions $H_k$, $R_k$ were introduced by Negrepontis [12 b].

1° A Boolean algebra $\underset{\sim}{A}$ satisfies the condition $H_k$ iff $\underset{\sim}{A}$ satisfies the following: Let $X, Y \subseteq A$, $X$ directed upward, $Y$ is directed downward, $0 \notin Y$, $1 \notin X$, $|X|+|Y|<k$, $X<Y$. Then there is an element $a \in A$ so that $X<a<Y$.

2° A Boolean algebra $\underset{\sim}{A}$ satisfies $R_k$ iff $\underset{\sim}{A}$ satisfies the following: Let $X, Y, Z \subseteq A$. Assume $X, Y$ satisfy the conditions in 1°, and $|X|+|Y|+|Z|<k$, $Z|<X$, $Y|<Z$. Then there is $a \in A$ so that $X<a<Y$, $a \| Z$.

It should be mentioned that the condition $R_k$ for arbitrary ordered sets, under the name of $k$-intercallation property, was introduced by Đ. Kurepa [10 b]. Kurepa considered this property in connection with $k$-universal ramified ordered sets.

The following theorem is given, for example, in [11 b].

Theorem 2.5. If a $BA$ $\underset{\sim}{A}$ satisfies $H_k$, then $\underset{\sim}{A}$ satisfies $R_k$. $\dashv$

Remark 2.6. Obviously, in definitions of conditions $H_k$, $R_k$, we may assume that $X$ is a proper ideal of $\underset{\sim}{A}$, and $Y$ a proper filter of $\underset{\sim}{A}$, where the condition $|X|+|Y|<k$ is replaced by $\tau(X)+\tau(Y)<k$, where $\tau(X)$ is the generating number of $X$ i. e. the least cardinal $\alpha$ so that there is a set of cardinality $\alpha$ which generates $X$. Then the separation principle $H_k$ can be stated also in the following form (and $R_k$ in a similar form): If $X$, $Y$ are proper ideals of $A$ so that $\tau(X)+\tau(Y)<k$ and $XY=0$, then there is $a \in A$ such that $X<a$, $Y<a'$. Of course, there are appropriate versions of $H_k$, $R_k$ in terms of filters.

Theorem 2.7. *A Boolean algebra $\underset{\sim}{A}$ is an atomless, $k$-saturated BA iff $\underset{\sim}{A}$ satisfies $H_k$.*

Proof: ($\Rightarrow$) Assume $A$ is atomless and $k$-saturated, and let $X$, $Y$ be as in definition of $H_k$. The set $\Gamma(c)=\{x<c:x\in X\}\cup\{c<y:y\in Y\}$ determinates a consistent type over $Th(A_{X\cup Y})$, so by $k$-saturation of $A$ there is an element $a\in A$ so that $A\models\Gamma(a)$, i. e. $X<a<Y$. ($\Leftarrow$) Now, assume $A$ satisfies $H_k$. Then $A$ is atomless because for any $a\in A$, $a>0$, by $H_k$ there is $b\in A$, $0<b<a$. Let $\Pi(x)$ be a maximal type over a subset $U$ of $A$, $|U|<k$. We show that $\Pi(x)$ is realized in $A$. $T^*$ allows elimination of quantifiers, thus $\Pi(x)$ is determinated by the set $\Pi'(x)$ of atomic and negatomic formulas contained in $\Pi(x)$. Let $C$ be the subalgebra generated by $U$. Then every atomic formula of $\Pi(x)$ is equivalent to a formula of the form $t(x)=0$, where $t(x)$ is a term over $L(T)\cup\{\underline{c}:c\in C\}$. Using the representation theorem on Boolean terms, there are $c_1, c_2\in C$ so that $c_1x'+c_2'x=0$. Thus, $t(x)=0\Leftrightarrow c_1\leqslant x\wedge x\leqslant c_2$, and $t(x)\neq 0\Leftrightarrow c_1\leqslant x\vee x\leqslant c_2$. Further, $c\leqslant x\Leftrightarrow c<x\vee c=x$, and because $\Pi(x)$ is a type, we have $(\underline{c}\leqslant x)\in\Pi(x)$ iff exactly one of $(\underline{c}<x)\in\Pi(x)$, $(\underline{c}=x)\in\Pi(x)$ holds.

We see that $\Pi(x)$ is an atomic type iff there is $c\in C$ so that $(x=\underline{c})\in\Pi(x)$. In that case, $\Pi(x)$ is trivially realized by $c$. Hence, assume $\Pi(x)$ is a nonprincipal type. Thus, for $X=\{c\in C:(\underline{c}\leqslant x)\in\Pi(x)\}$, $Y=\{c\in C:(x\leqslant\underline{c})\in\Pi(x)\}$, $Z=\{c\in C:(x\,\|\,\underline{c})\in\Pi(x)\}$ we have that $X$, $Y$, $Z$ satisfy conditions in $R_k$. Since $A$ satisfies $H_k$, by T. 2.5 $A$ also satisfies $R_k$, thus there is an element $a\in A$ so that $X<a<Y$, $a\,\|\,Z$. Hence, a satisfies $\Pi'(x)$, and thus $\Pi(x)$. $\dashv$

Corollary 2.8. *Let $A$ be a BA. Then the following are equivalent (for the equivalence of* 2° *and* 3° *also cf.* [4], *§ 6):*

1° *$A$ is a saturated atomless BA.* 2° *$\underset{\sim}{A}$ satisfies $H_k$, where $k=|A|$.* 3° *$\underset{\sim}{A}$ is a full BA.*

Corollary 2.9. (cf. [4], § 6) *Let $\underset{\sim}{A}, \underset{\sim}{B}$ be BA's. If $|A|=|B|=k$, and $\underset{\sim}{A}, \underset{\sim}{B}$ satisfy $H_k$, then $\underset{\sim}{A}\cong\underset{\sim}{B}$.*

Proof: $A, B$ are atomless, thus by completeness of $T^*$ $A\equiv B$. By C. 2.8. $\underset{\sim}{A}, \underset{\sim}{B}$ are saturated, so by uniqueness of saturated models $\underset{\sim}{A}\cong\underset{\sim}{B}$. $\dashv$

Now we give another characterization of atomless, $k$-saturated BA's. Before we do that, we list some definitions and theorems.

Definition 2.10. 1° Let $\mathfrak{M}$ be a class of models of a language $L$, and $k$ an infinite cardinal. A model $\mathfrak{A}\in\mathfrak{M}$ is called $k$-injective in $\mathfrak{M}$ iff for any $\mathfrak{B}, \mathfrak{C}\in\mathfrak{M}$, so that $|B|<k$, $|C|\leqslant k$, and any embeddings $f:\mathfrak{B}\to\mathfrak{A}$, $g:\mathfrak{B}\to\mathfrak{C}$, there is an embedding $h:\mathfrak{C}\to\mathfrak{A}$ so that $hg=f$ (also cf. [19]).

2° A model $\mathfrak{A}\in\mathfrak{M}$ is injective iff it is $|A|$-injective.

3° $\mathfrak{A}$ is elementary ($k-$) injective iff all above embeddings are elementary.

The following theorems are given in [15]. In their formulation it is assumed that languages in which theories are formulated are countable.

Theorem 2.11. Let $k$ be uncountable. Then $\mathfrak{A}$ is $k$-saturated iff $\mathfrak{A}$ is elementary $k$-injective. $\dashv$

Theorem 2.12. (L. Blum) Let $\Gamma$ and $\Gamma^*$ be theories in the same countable language, and suppose that $\Gamma \subseteq \Gamma^*$, $\Gamma$ is universal, and every model of $\Gamma$ can be extended to some model $\Gamma^*$. Then $\Gamma^*$ is the model-completion of $\Gamma$ iff every diagram of the following sort can be completed as shown:

$$\begin{tikzcd} \mathfrak{B}^* & \\ \mathfrak{B} \arrow[u] \arrow[r, phantom, "\subseteq"] & \mathfrak{B}(c) \arrow[ul] \end{tikzcd} \qquad \begin{array}{l} \mathfrak{B},\ \mathfrak{B}(c) \models \Gamma \\ \mathfrak{B}^* \models \Gamma^* \\ \mathfrak{B}^* \text{ is } |B|^+ \text{— saturated. } \dashv \end{array}$$

In the following theorem another description of atomless $k$-saturated BA's is given.

Theorem 2.13. *Let $B$ be a BA, and $k$ an infinite cardinal. Then the following are equivalent:*

1° *$B$ satisfies $H_k$.*

2° *For every BA $A$, $|A|<k$, every embedding $f: A \to B$ can be extended to an embedding $g: A(c) \to B$.*

3° *$B$ is $k$-injective in the class of all BA's.*

Proof: (1° $\Rightarrow$ 2°) Assume $B$ satisfies $H_k$, and $|A|<k$. First assume $k>\omega$. Then by T. 2.7. $B$ is $k$-saturated, hence conditions of T. 2.12. are satisfied. By the same theorem the asertion holds. If $k=\omega$, the assertion holds by L. 1.3.

(2° $\Rightarrow$ 3°) Let $f: A \xrightarrow[1-1]{} B$, $g: A \xrightarrow[1-1]{} C$, where $|A|<k$, $|C| \leqslant k$, and $C-g(A)=\{c_\alpha : \alpha<k\}$. Define the chain of BA's $\langle C_\alpha : \alpha<k \rangle$ by $C_0=g(A)$, $C_{\alpha+1}=$ Boolean subalgebra of $C$ generated by $C_\alpha \cup \{c_\alpha\}$, and if $\alpha$ is limit then $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$. Then $h: C \to B$ is defined by $k$ consecutive uses of 2°.

(3° $\Rightarrow$ 1°) Assume $B$ is $k$-injective in the class of all BA's. First we prove that $B$ is atomless. Let $b \in B$, $b, b' \neq 0$. Consider Boolean subalgebra $\{0, 1, b, b'\}$ of $B$. Obviously this BA is isomorphic to the field of sets $\{\varnothing, \{0, 1\}, \{2\}, \{0, 1, 2\}\}$, and an isomorphism is given by $g(b)=\{0, 1\}$, $g(b')=\{2\}$. Thus, there is an embedding $h$ of the field $S(\{0, 1, 2\})$ into $B$ with $hg(b)=b$. Then $0<h(\{0\})<b$. Hence $B$ is atomless. If $k=\omega$, $B$ satisfies $H_k$ since $B$ is atomless. So assume $k>\omega$. Since $\mathfrak{M}(T^*) \subseteq \mathfrak{M}(T)$, $B$ is also $k$-injective in $\mathfrak{M}(T^*)$, $T^*$ is model-complete, thus all embeddings in question are elementary. Hence, $B$ is elementary $k$-injective, thus by T. 2.11. $B$ is $k$-saturated. $\dashv$

Corollary 2.14. *If BA's $A, B$ are injective and $|A|=|B|$, then $A \cong B$.*

Proof: By the previous theorem and uniqueness of saturated models. $\dashv$

Proposition 2.15. *If $k$ is infinite and $B$ satisfies $H_k$ then $B$ is a $k$-universal model in $\mathfrak{M}(T)$.*

Now we give some other properties of $k$-saturated atomless Boolean algebras.

Definition 2.16. 1° Let $A$ be a $BA$. A nonempty subset $S \subseteq A$ is a cellular family iff $0 \notin S$, and $\forall x, y \in S$ $(x \neq y \Rightarrow xy = 0)$.

2° The cellular number of a $BA$ $A$, $\mathrm{cel}(A)$, is $\sup \{|S| : S \subseteq A,\ S$ is a cellular family in $A\}$ (cf. [10 a] p. 131).

Proposition 2.17. *Assume $k$ is infinite and $B$ satisfies $H_k$. Then $cel(B) \geqslant k$.*

Proof: $B$ is an infinite $BA$, thus there is an infinite cellular subset $S \subseteq B$. Hence $\mathrm{cel}(B) \geqslant \omega$. Now, let $S$ be any infinite cellular family of $B$. By Zorn's Lemma there is a maximal cellular family $S' \supseteq S$. Assume $|S'| < k$. Then $\Gamma(x) = \{xs = 0 : s \in S'\} \cup \{x \neq 0\} \cup \{x \neq s : s \in S'\}$ is finitely consistent, thus by $k$-saturation of $B$ it is realized in $B$ by an element $b$. Hence, $S'$ is not a maximal cellular family, what is absurd. $\dashv$

Lemma 2.18. *Let $B$ be a $BA$, and $S \subseteq B$ a cellular family. Then $S$ can be collapsed into a set of atoms, i. e. there is an atomic Boolean algebra $A$ and a homomorphism $f : B \xrightarrow{\text{onto}} A$ so that the set of atoms of $A$ is $\{f(x) : x \in S\}$, and for $x, y \in S$ if $x \neq y$ then $f(x) \neq f(y)$.*

Proof: Let $a \in B$, $a \neq 0$. Then $B_a = (B_a, +, \cdot, {}'^a, \leqslant, 0, a)$, where $B_a = \{x \in B : x \leqslant a\}$, $x'^a = ax'$, is also a Boolean algebra. Let for each $a \in S$ $I_a$ be a maximal ideal in $B_a$. Let $I$ be the ideal of $B$ generated by $\bigcup_{a \in S} I_a$, $B' = B/I$, and $k : B \to B'$ the canonical homomorphism. Further, let $J = \{b \in B' : \forall a \in S\, (bk(a) = 0)\}$, It is easy to see that $J$ is an ideal, so let $A = B'/J$, and $h : B' \to A$ the canonical homomorphism. Then $f = hk$ is a homomorphism from $B$ onto $A$, and the set of atoms of $A$ is $\{f(x) : x \in S\}$. $\dashv$

Proposition 2.19. *Let $B$ be $k^+$-saturated atomless $BA$, where $k$ is an infinite cardinal. Then the Boolean algebra $S(k)$ of subsets of $k$ is a homomorphic image of $B$.*

Proof: Since $B$ is $k^+$-saturated, by P. 2.17. there is a cellular family $S \subseteq B$ of cardinality $k$. By the previous lemma there is an atomic $BA$ $C$, $h : B \xrightarrow{\text{onto}} C$, so that the set of atoms of $C$ is $A = \{h(x) : x \in S\}$. Now we prove that $C$ is complete. Let $X \subseteq A$ and $Y = A - X$. Further, let $U = \{x \in S : h(x) \in X\}$, $V = \{x \in S : h(x) \in Y\}$, and $\Gamma(x) = \{u \leqslant x : u \in U\} \cup \{xv = 0 : v \in V\}$. $\Gamma(x)$ is finitely consistent, $U \cup V = S$, and $|S| \leqslant k$. Thus, by $k^+$-saturation of $B$, $\Gamma(x)$ is realized in $B$, say by $a$. Hence, $\forall u \in U\, (h(u) \leqslant h(a))$, $\forall v \in V\, (h(v)\, h(a) = 0)$ so $h(a) = \sup X$. Therefore, $C \cong S(k)$. $\dashv$

Corollary 2.20. *Assume $B$ is $k^+$-saturated. Then $|B| \geqslant 2^k$.* $\dashv$

Proposition 2.21. *Let $k$ be an uncountable cardinal, $\underset{\sim}{B}$ a $k$-saturated atomless BA. If an atomic BA $\underset{\sim}{C}$ with countable set of atoms is a homomorphic image of $\underset{\sim}{B}$, then $\underset{\sim}{C}$ is complete, i. e. $\underset{\sim}{C} \cong S(\omega)$.*

Proof: If $C$ is finite there is nothing to prove. So assume $\underset{\sim}{C}$ infinite, and let $A=\{a_n : n\in\omega\}$ be the set of atoms of $C$. Now, we prove that every subset $S\subseteq A$ has the supremum. If $S$ or $A-S$ is finite then the assertion is obvious, so assume $|S|$, $|A-S|=\omega$, say $S=\{b_n : n\in\omega\}$, $A-S=\{c_n : n\in\omega\}$. Let for each $b_i$, $c_i$ $\beta_i$, $\gamma_i\in B$ so that $h(\beta_i)=b_i$, $h(\gamma_i)=c_i$, and $u_n=\sum\limits_{i\leqslant n}\beta_i$, $v_n=\sum\limits_{i\leqslant n}\gamma_i$, $i\in\omega$. Then $u_0\leqslant u_1\leqslant\cdots$ and $v_0\leqslant v_1\leqslant\ldots$, thus $u_0v_0'+u_1v_1'+\cdots+u_nv_n'\leqslant\leqslant(u_0+v_0')(u_1+v_1')\cdots(u_n+v_n')$ for each $n\in\omega$. Since $uz'+vz\leqslant uv \Leftrightarrow uv'\leqslant z\leqslant\leqslant u+v'$, we have $u_0z'+v_0z\leqslant u_0v_0\wedge\cdots\wedge u_nz'+v_nz\leqslant u_nv_n \Leftrightarrow$

$$u_0v_0'+\cdots+u_nv_n'\leqslant z\leqslant(u_0+v_0')\cdots(u_n+v_n').$$

Therefore, $\Gamma(z)=\{u_n z'+v_n z\leqslant u_n v_n : n\in\omega\}$ is finitely consistent, so there is $b\in B$ which realizes $\Gamma(z)$. Let $c=h(b)$. Then for each $n\in\omega$ $(b_0+\cdots+b_n)c'+ +(c_0+\cdots+c_n)c=0$, i. e. for all $x\in S$ $x\leqslant c$, and for all $x\in A-S$ $x\leqslant c'$. Thus $c=\sup S$. $\dashv$

Corollary 2.22. *Let $k$ be an uncountable cardinal and $\underset{\sim}{B}$ $k$-saturated, atomless BA. If a BA $A$ is an infinite homomorphic image of $\underset{\sim}{B}$, then $S(\omega)$ is a homomorphic image of $\underset{\sim}{A}$ (and thus $|A|\geqslant 2^{\omega}$).* $\dashv$

We can say something more about the cardinality of $k$-saturated Boolean algebras.

Definition 2.23. The saturation number of a model $\mathfrak{A}$, sat$(\mathfrak{A})$, is $\sup\{k : \mathfrak{A}$ is $k$-saturated$\}$.

Theorem 2.24. *Let $\underset{\sim}{A}$ be an atomless BA. Then sat$(\underset{\sim}{A})$ is a regular cardinal.*

Proof: It is obvious that it suffices to prove the following: If $k$ is a singular cardinal and if $A$ satisfies $H_k$, then $\underset{\sim}{A}$ satisfies $H_{k^+}$. So suppose $k$ is singular and $\underset{\sim}{A}$ satisfies $H_k$. By R. 2.6. it suffices to prove that for any two proper ideals $I$, $J$ with $IJ=0$, $\tau(I)+\tau(J)\leqslant k$, there is an element $a\in B$ so that $I<a$, $J<a'$. Obviously, interesting cases are when $\tau(I)=k$, or $\tau(J)=k$. So assume $\tau(I)=\tau(J)=k$, and let $\lambda=cf(k)$. Thus, $\lambda<k$, and for a generating set $S$ of the ideal $I$, there is a sequence of sets $S_\alpha$ $(\alpha<\lambda)$ so that $S=\bigcup\limits_{\alpha<\lambda}S_\alpha$, and if $\alpha<\beta<\lambda$ then $S_\alpha\underset{\neq}{\subset}S_\beta$, and $|S_\alpha|\leqslant|S_\beta|<k$. Let $I_\alpha$ be the ideal generated by $S_\alpha$. Thus, $I=\bigcup\limits_{\alpha<\lambda}I_\alpha$, and $\alpha<\beta<\lambda$ implies $I_\alpha\subset I_\beta$ and $\tau(I_\alpha)\leqslant\tau(I_\beta)$. In a similar way, we can represent the ideal $J$ as a union of an ascending chain of ideals $J_\alpha$, $\alpha<\lambda$, with similar properties. As $IJ=0$, we have $I_\alpha J_\beta=0$ for all $\alpha$, $\beta<\lambda$, thus by $H_k$ there are elements $a_\alpha$ with $I_\alpha<a_\alpha$, $J_\alpha<a_\alpha'$.

We construct two sequences $(b_\alpha : \alpha<\lambda)$, $(c_\alpha : \alpha<\lambda)$ of elements of $\underset{\sim}{A}$ which satisfy: 1° $I_\alpha\leqslant b_\alpha$, $J_\alpha\leqslant c_\alpha$, 2° For $\alpha\leqslant\beta<\lambda$, $b_\alpha\leqslant b_\beta<1$, $c_\alpha\leqslant c_\beta<1$, 3° For $\alpha$, $\beta<\lambda$ $b_\alpha c_\beta=0$, 4° For all $\alpha<\lambda$ $b_\alpha\leqslant F_\alpha$, $c_\alpha\leqslant G_\alpha$, where $F_\alpha$ is the filter generated by

$\{a_\rho : \lambda > \rho \geqslant \alpha\}$, and $G_\alpha$ is the filter generated by $\{a'_\rho : \lambda > \rho \geqslant \alpha\}$. Now assume we have constructed $b_\beta$, $c_\beta$ for $\beta < \alpha$. Let $I'$ be the ideal generated by $I_\alpha \cup \{b_\beta : \beta < \alpha\}$. Then $\tau(I') < k$ and $I' < F_\alpha$. Thus, there is an element $x \in A$ so that $I' \leqslant x \leqslant F_\alpha$. Let $b_\alpha = x$, $c_\alpha$ is constructed similarly. It is easy to see that $b_\alpha$, $c_\alpha$ satisfy conditions $1^\circ - 4^\circ$. Let $I_1, J_1$ be ideals of $\underset{\sim}{A}$ generated respectively by $\{b_\alpha : \alpha < \lambda\}$. Then $I_1 J_1 = 0$ and $\tau(I_1) + \tau(J_1) \leqslant \lambda < k$, so by $H_k$ there is $a \in B$ so that $I_1 < a$, $J_1 < a'$. Since $I \subseteq I_1$, $J \subseteq J_1$, $a$ also separates $I$ and $J$.

The case $\tau(I) = k$, $\tau(J) < k$ is done in a similar way. $\dashv$

Corollary 2.25. *Let $\underset{\sim}{A}$ be an atomless BA and $k = sat(\underset{\sim}{A})$. Then*

$$|A| \geqslant k^{\underset{\smile}{k}} \quad \left(k^{\underset{\smile}{k}} = \sum_{\mu < k} k^\mu\right).$$

Proof: Assume $k = \lambda^+$. Then by C. 2.20 $|A| \geqslant 2^\lambda = k^{\underset{\smile}{k}}$.

Assume $k$ is a limit cardinal. By C. 2.20. $|A| \geqslant \sup_{\mu < k} 2^\mu = 2^{\underset{\smile}{k}}$. By the previous theorem $k$ is regular, thus $2^{\underset{\smile}{k}} = k^{\underset{\smile}{k}}$, hence $|A| \geqslant k^{\underset{\smile}{k}}$. $\dashv$

Corollary 2.26. (cf. [4], T. 6.12.) $1^\circ$ *Assume $\underset{\sim}{A}$ is a saturated BA of cardinality $k$. Then $k = k^{\underset{\smile}{k}}$.*

$2^\circ$ *If $k = k^{\underset{\smile}{k}}$ then there is a saturated BA of cardinality $k$.*

Proof: $1^\circ$ Follows from the previous corollary.

$2^\circ$ By the theorem on existence of saturated models. $\dashv$

Atomless Boolean algebras which are $\omega_1$-saturated have got the attention of several mathematicians, starting with Hausdorff (cf. [9]). For other historical remarks one can consult [4], § 6.

Proposition 2.27. An atomless *BA* $\underset{\sim}{A}$ is $\omega_1$-saturated iff for any two nonempty chains $R, S$ of $A$ with $|R| + |S| \leqslant \omega$, $R < S$, there is $a \in A$ such that $R < a < S$.

Proof: Let $X$, $Y$ be as in definition of $H_{\omega_1}$ and assume $X = \{a_n : n \in \omega\}$, $Y = \{b_n : n \in \omega\}$. Define $r_n = \sum_{k \leqslant n} a_k$, $s_n = \sum_{k \leqslant n} b_k$. $\dashv$

The following example essentially belongs to H. J. Keisler; he proved the clause $2^\circ$ in T. 2.13. (cf. [9]). We show that this *BA* satisfies $H_{\omega_1}$, so we believe that this proof is simpler.

Example 2.28. Let $\{\underset{\sim}{A}_n : n \in \omega\}$ be a countable set of atomless Boolean algebras, and $D$ the filter of cofinite subsets of $\omega$. Then the reduced product $\underset{\sim}{B} = \prod_{n \in \omega} \underset{\sim}{A}_n / D$ is an atomless $\omega_1$-saturated *BA*.

Proof: Let $\underset{\sim}{A}=\prod_{n\in\omega}\underset{\sim}{A_n}$. We observe several facts:

1° $B=A/I$, where $I=\{f\in A: f(n)=0$ for all $n$ except finitely many$\}$.

2° $f/I=g/I$ iff $f(n)\neq g(n)$ for at most finitely many $n\in\omega$.

3° $f/I\geqslant g/I$ iff there is $h\in A$ so that $g/I=h/I$ and $h\leqslant f$ (observe that $h\leqslant f$ iff $\forall n\in\omega\,(h(n)\leqslant f(n))$).

4° $g/I<f/I$ iff $\{n\in\omega: g(n)\nleq f(n)\}$ is finite, $\{n\in\omega: g(n)<f(n)\}$ is infinite.

First, let us prove that $B$ is atomless. So assume $f/I>0$. Thus, $S=\{n\in\omega: f(n)>0\}$ is infinite. Let $S=S_1\cup S_2$, $S_1\cap S_2=\varnothing$, $S_1, S_2$ are infinite. Define $h$ by $h(n)=f(n)$ if $n\in S_1$, $h(n)=0$ otherwise. Then $f/I>h/I>0$.

According to the last proposition it suffices to show the separation principle $H_{\omega_1}$ for chains of $A$. We distinguish two cases. The first is when the left chain has the greatest element, or the right chain has the smallest element. The second case is when the left chain has no greatest element, neither the right chain has the smallest element.

So assume the first case: $0<\cdots<f_1/I<f_0/I$. By 3° we may assume that $\cdots\leqq f_1\leqq f_0$. Since for each $k\in\omega$ the set $\{n\in\omega: f_k(n)>0\}$ is infinite, there is a sequence $(n_k: k\in\omega)$ so that for all $k\in\omega$ $f_k(n_k)>0$, $n_k<n_{k+1}$. Define $h$ as follows: $h(n_k)=f_k(n_k)$, $h(n)=0$ for $n\notin\{n_k:\in\omega\}$. Then $0<h/I<f_n/I$ for all $n\in\omega$.

Now, assume the second case: $f_0/I>f_1/I>\cdots>g_1/I>g_0/I$. By 3° we may assume $f_0\geqslant f_1\geqslant\cdots$, and $\cdots\geqslant g_1\geqslant g_0$. Let $s_k$ be the least $m\in\omega$ such that $\forall n\geqslant m\,(g_k(n)\leqslant f_k(n))\cdot s_k$ exists by 4°. Obviously, $s_k$ is an ascending sequence. We consider two cases: (1) There is $m_0$ so that $\forall n\geqslant m_0\,(s_n=s_{m_0})$. Let $n_0=\max(m_0, s_{m_0})$ (remark that for $n\geqslant n_0\, g_n(n)\leqslant f_n(n)$). Let $h$ be defined as follows: for $n<n_0\, h(n)=0$; if $n\geqslant n_0$ then $h(n)=g_n(n)$. Then it is easily seen that $f_0/I>f_1/I>\cdots>h/I>\cdots>g_1/I>g_0/I$. (2) The sequence $s=(s_k: k\in\omega)$ is cofinal in $\omega$. Thus, $s$ has a strictly increasing subsequence. The corresponding subsequences in $f_n/I, g_n/I$ are respectively coinitial and cofinal, so an element which separates these subsequences also separates $f_n/I$ and $g_n/I$. Thus, without loss of generallity we may assume that $s$ itself is strictly increasing. Let us define $h$ as follows: for $n<s_0\, h(n)=0$; if $k\in\omega$ and $s_k\leqslant n<s_{k+1}$ then $h(n)=g_k(n)$. It is easily seen that $f_n/I>h/I>g_m/I$ for all $m, n\in\omega$. ⊣

Corollary 2.29. 1° (H. J. Keisler) Assume Continuum Hypothesis ($CH$). If for all $i\in\omega$ $A_i, B_i$ are Boolean algebras of cardinality at most $\omega_1$, then $\prod_{i\in\omega}A_i/D\cong\prod_{i\in\omega}B_i/D$.

2° $S(\omega)/S_\omega(\omega)$ (the field of subsets of $\omega$ modulo the ideal of finite subsets) is $\omega_1$-saturated $BA$.

3° Assume $CH$. $S(\omega)/S_\omega(\omega)$ is a saturated atomless $BA$ of cardinality $2^\omega$.

Proof: 1° Observe that $\omega_1^\omega\geqslant|\prod_{i\in\omega}A_i/D|=|\prod_{i\in\omega}B_i/D|\geqslant 2^\omega$, and $\omega_1^\omega=2^\omega=\omega_1$. Thus, the assertion hods by the previous example and the uniqueness of saturated models.

2° $S(\omega)/S_\omega(\omega)\cong 2^\omega/D$.

3° By the previous example and 2°. ⊣

## 3. Boolean algebras with ultrafilters.

In this part, Boolean algebras together with $\lambda$ distinct ultrafilters are considered. First, the case $\lambda$ is finite is investigated, then $\lambda$ arbitrary. Once again, let us remember that $T_\lambda$ denotes the theory of these models. If $\mathfrak{A}$ is a model of $T_\lambda$, then $\mathfrak{A}=(A, U_\alpha)_{\alpha<\lambda}$, where $A$ is a $BA$, and $U_\alpha\,(\alpha<\lambda)$ an ultrafilter over $A$. Ultrafilters are denoted also by $V, P, Q$ etc. If $P$ is an ultrafilter over $A$, then $P^C=A-P=\{x': x\in P\}$, i.e. dual maximal ideal. Remark that $x\in P^C$ iff $x'\in P$.

Proposition 3.1. Let $\lambda$ be finite. Then $T_\lambda$ is $\omega$-categorical.

Proof: First we prove the assertion for $\lambda=1$. Let $\Omega$ be a countable free $BA$ with $\omega$ free generators $a_0, a_1, \ldots$ . $\Omega$ is an atomless $BA$, thus, by T. 1.4. every atomless countable $BA$ is isomorphic to $\Omega$. Hence, it suffices to show that for any two ultrafilters $U, V$ over $\Omega$, $(\Omega, U)\cong(\Omega, V)$. As $U$ is an ultrafilter, $U$ decides for any $a_i$ if $a_i\in U$, or $a_i'\in U$, and similarly holds for $V$. Hence, there are $\alpha, \beta\in 2^\omega$ so that $a_i^{\alpha(i)}\in U$, $a_i^{\beta(i)}\in V$, $i\in\omega$ (here $c^0=c'$, $c^1=c$), and for the map $f$ defined by $f(a_i^{\alpha(i)})=a_i^{\beta(i)}$, $i\in\omega$, there is a (unique) isomorphism $h:\Omega\to\Omega$, $f\subseteq h$. Then, also, $h:(\Omega, U)\cong(\Omega, V)$.

Now, let $\lambda=n+1$ be arbitrary and finite, $n\geqslant 1$. Further, assume $\mathfrak{A}=(A, P_0, \ldots, P_n)$, $\mathfrak{B}=(B, Q_0, \ldots, Q_n)$, $\mathfrak{A}, \mathfrak{B}\models T_\lambda^*$, $|A|, |B|=\omega$.

Claim There are $a_i\in P_i$, $i=0, 1, \ldots, n$, so that $a_i a_j=0$ for $i\neq j$, $\sum\limits_{i\leqslant n} a_i=1$.

Proof of claim: Since $U_i\neq U_j$ for $0\leqslant i<j\leqslant n$, there are elements $c_{ij}\in U_i$, $c_{ij}'\in U_j$ for $0\leqslant i<j\leqslant n$. Let $b_k=\prod\limits_{i<k} c_{ik}'\cdot\prod\limits_{k<j\leqslant n} c_{kj}$ for $k\leqslant n$, where the empty product is taken to be 1. Then $b_k\in U_k$, $b_k b_m=0$ for $k\neq m$. Let $a_0=b_0+\prod\limits_{k\leqslant n} b_k'$, and $a_k=b_k$ for $k\geqslant 1$. Then the elements $a_k$ satisfy the required conditions.

Remark that this claim holds for any $\mathfrak{A}\models T_n$.

Now we return to the proof of the proposition. By the claim, there is also a sequence $(b_k:k\leqslant n)$ so that $b_k\in Q_k$, $b_k b_m=0$ for $k\neq m$, $\sum\limits_{k\leqslant n} b_k=1$. Observe that $A_{a_i}\cap P_i$ is an ultrafilter over $\underset{\sim}{A}_{a_i}$, and $\underset{\sim}{A}_{a_i}$ is an atomless $BA$. Thus, by the first part of the proof, there are isomorphisms

$$f_i:(\underset{\sim}{A}_{a_i}, A_{a_i}\cap P_i)\cong(\underset{\sim}{B}_{b_i}, B_{b_i}\cap Q_i)$$

Then the map $f:A\to B$ defined by $f(x)=\sum\limits_{i\leqslant n} f_i(a_i x)$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. $\dashv$

Bu Lindström's theorem and Łoš-Vaught test we have:

Corollary 3.2. *$T_\lambda^*$ is model-complete for finite $\lambda$.*

Corollary 3.3. *$T_\lambda^*$ is complete for finite $\lambda$.*

Definition 3.4. Let $\underset{\sim}{A}$ be a $k$-saturated $BA$, $U$ an ultrafilter, and $J$ a maximal ideal over $\underset{\sim}{A}$. Then:

1° $U$ is a $k$-saturated ultrafilter over $\underset{\sim}{A}$ iff $(\underset{\sim}{A}, U)$ is a $k$-saturated model.

2° $J$ is a $k$-saturated ideal over $\underset{\sim}{A}$ iff $(\underset{\sim}{A}, J)$ is a $k$-saturated model.

3° $U$ is a $k$-directed iff for any subset $S\subseteq U$, $|S|<k$, there is an element $a\in U$ so that $a<S$.

4° $J$ is $k$-directed iff for every subset $S\subseteq J$, $|S|<k$, there is an element $a\in J$ such that $S<a$.

These notions are introduced in order to describe saturated models of $T_\lambda$. It will appear that $k$-saturated models of $T_\lambda^*$ are exactly those models $(\underset{\sim}{A}, P_\alpha)_{\alpha<\lambda}$ in which the ultrafilters $P_\alpha$ are $k$-directed.

Proposition 3.5. *Let $\underset{\sim}{A}$ be a $k$-saturated BA. Then:*

1° *An ultrafilter (maximal ideal) $S$ over $\underset{\sim}{A}$ is $k$-saturated iff $S^C$ is a $k$-saturated maximal ideal (ultrafilter) over $\underset{\sim}{A}$.*

2° *An ultrafilter (maximal ideal) $S$ over $\underset{\sim}{A}$ is $k$-directed iff $S^C$ is a $k$-directed ideal (ultrafilter).*

Proof: Observe that $(A, +, \cdot, ', 0, 1, \leq) \cong (A, \cdot, +, ', 1, 0, \geqslant)$. ⊣

Lemma 3.6. *Let a BA $\underset{\sim}{A}$ satisfy the condition $H_k$, assume $k$ is an infinite cardinal, and $P$ is a $\tilde{k}$-directed ultrafilter over $A$. Further, suppose $X, Y, Z \subseteq A$ so that $|X|+|Y|+|Z|<k$, $X<Y$, $1 \notin X$, $0 \notin \tilde{Y}$, $X$ is upward and $Y$ is downward directed, $Z \,|\!< X$, $Y \,|\!< Z$. Then: 1° If $Y \subseteq P$ then there is $p \in P$ such that $X<p<Y$, $p \,\|\, Z$, 2° If $X \cap P = \varnothing$ then there is $p \in A$ such that $p' \in P$, and $X<p<Y$, $p \,\|\, Z$.*

Proof: First, let us observe that by T. 2.5. $\underset{\sim}{A}$ satisfies the condition $R_k$. Further, we need the following assertion.

Claim: For any subsets $X$, $Y$ of $A$ such that $X<Y$, $|X|+|Y|<k$, $X$ is upward directed, $Y$ is downward directed, $Y \subseteq P$, there is an element $a \in P$ which satisfies $X<a<Y$.

Proof of the claim: By $H_k$ there is $g \in A$, $X<g<Y$. If $X \cap P \neq \varnothing$, then obviously $g \in P$. So assume $X \cap P = \varnothing$, and $g \notin P$. Thus, $g' \in P$. Then the set $S=\{yg' : y \in Y\}$ satisfies $S \subseteq P$, and $|S|<k$. As $P$ is $k$-directed, there is $c \in P$, $c<S$. Let $a=g+c$. Thus $a \in P$ and $X<a \leq Y$. If for some $y_0 \in Y$ $a=y_0$, then $y_0=g+c$, i. e. $y_0 g'=cg'$, what contradicts to $y_0 g'>c$. Hence $X<a<Y$.

Dualizing this proof in $(A, \cdot, +, ', \geqslant, 1, 0)$ we have also: For $X$, $Y$ as above, but with $X \cap P = \varnothing$, there is $a \in A$ such that $X<a<Y$, and $a' \in P$.

Now, we prove 1°. Thus, assume $Y \subseteq P$. By $R_k$ there is $a \in A$ such that $X<a<Y$, $a \,\|\, Z$. By the claim there is $b \in P$ such that $a<b<Y$. Observe that there is no $z \in Z$ so that $b \leqslant z$. If $a \in P$ we are done, so assume $a \notin P$. Let $Z=\{c_\beta : \beta<\alpha\}$ $(\alpha<k)$. A descending sequence $(b_\beta : \beta<\alpha)$ is defined inductively so that: (1) $b_\beta \in P$, (2) $a<b_\beta$, (3) $b_\beta \,\|\, \{c_\gamma : \gamma<\beta\}$, (4) $b_\beta \not\leq c_\gamma$. By the claim there is $d \in A$ so that $0<d<c_0 a'$, $d \notin P$. Let $b_0=bd'$.

*The case* $\beta=\gamma+1$. Let $b_\beta$ be constructed from $a$, $b_\gamma$, $c_\beta$ as $b_0$ was from $a$, $b$, $c_0$.

*The case $\beta$ is limit.* Let $u \in A$ so that $a<u<\{b_\rho : \rho<\beta\}$. Then $b_\beta$ is constructed from $a$, $u$, $c_\beta$ as $b_0$ was from $a$, $b$, $c_0$.

Now, let $x \in U$, $a<x<\{b_\beta : \beta<\alpha\}$, and set $p=x$.

The assertion 2° is obtained dualizing the above proof. ⊣

Lemma 3.7. *Let a BA $\underset{\sim}{A}$ satisfy $H_k$, where $k$ is an infinite cardinal. Suppose $\lambda$, $\mu$ are cardinals so that $\lambda, \mu<k$, and $(P_\alpha:\alpha<\lambda)$, $(Q_\alpha:\alpha<\mu)$ are two sequences of k-directed ultrafilters over $\underset{\sim}{A}$ so that for all $\alpha<\lambda$, $\beta<\mu$ $P_\alpha\neq Q_\beta$, Then $(\bigcap_{\alpha<\lambda} P_\alpha)\cap(\bigcap_{\beta<\mu} Q_\beta^c)\neq\varnothing$. We are assuming $\lambda, \mu\geqslant 1$.*

Proof. First we prove the following assertion.

Claim: $P\cap(\bigcap_{\beta<\mu} Q_\beta^c)\neq\varnothing$.

Proof of the claim: We construct a decreasing sequence $a_\beta$, $\beta<\mu$ so that $a_\beta\in P$ and $a_\beta\in Q_\beta^c$.

*The case* $\beta=0$. Since $P\neq Q_0$ there is $a_0\in P$ so that $a_0\in Q_0^c$.

*The case* $\beta=\gamma+1$. Assume $a_\rho$ has been defined for $\rho\leqslant\gamma$. As $P\neq Q_\beta$, there is $c\in P$ such that $c\in Q_\beta^c$. Thus, $ca_\gamma\in P$. Since $P$ is not principal, there is $x\in P$, $x<ca_\gamma$. Let $a_\beta=x$. Observe that $Q_\beta^c$ is an ideal, and $c\in Q_\beta^c$, hence $a_\beta\in Q_\beta^c$.

*The limit case.* Suppose $\beta$ is limit. Since $P$ is $k$-directed, there is $x\in P$ such that $x<a_\rho$ for all $\rho<\beta$. Let $a_\beta=x$.

Hence, the claim is proved.

Now, we proceed to the proof of the lemma. For that, we construct an increasing sequence $(b_\alpha:\alpha<\lambda)$ such that $b_\alpha\in P_\alpha$. $b_\alpha\in\bigcap_{\beta<\mu} Q_\beta^c$.

If $\alpha=0$, by the claim there is $b_0\in P_0$ such that $b_0\in\bigcap_{\beta<\mu} Q^c$.

If $\alpha=\gamma+1$, by the claim there is $x\in P_\alpha\cap(\bigcap_{\beta<\mu} Q_\beta^c)$. Then $x+b_\gamma\in P_\alpha$. By the inductive hypothesis, $b_\gamma\in\bigcap_{\beta<\mu} Q_\beta^c$, thus $x+b_\gamma\in\bigcap_{\beta<\mu} Q_\beta^c$ (observe that $\bigcap_{\beta<\mu} Q_\beta^c$ is an ideal). So let $b_\alpha=x+b_\gamma$.

Assume $\alpha$ is limit. We have $b_\gamma\in\bigcap_{\beta<\mu} Q_\beta^c$ for all $\gamma<\alpha$. Since $Q^c$ 's $k$-directed, there is $d_\beta\in Q_\beta^c$ such that for all $\gamma<\alpha$ $b_\gamma<d_\beta$. The set $\{b_\gamma:\gamma<\alpha\}$ is directed upward, thus by the condition $H_k$ there is $x\in A$ such that $b_\gamma\leqslant x\leqslant d_\beta$ for all $\gamma<\alpha$, $\beta<\mu$. Hence, $x\in\bigcap_{\beta<\mu} Q_\beta^c$, and $x\in P_\alpha$, so let $b_\alpha=x$.

Now, first assume $\lambda$ is finite. Define $c=b_\lambda$. Then

$$c\in(\bigcap_{\alpha<\lambda} P_\alpha)\cap(\bigcap_{\alpha<\mu} Q_\alpha^c).$$

Suppose $\lambda$ is infinite. Then we can construct $c$ as $b_\alpha$ was constructed for limit $\alpha$. Again, $c\in(\bigcap_{\alpha<\lambda} P)\cap(\bigcap_{\alpha<\mu} Q_\alpha^c)$. $\dashv$

Lemma 3.8. *Let a BA $\underset{\sim}{A}$ satisfy $H_k$ ($k$ is an infinite cardinal), and $(P_\alpha:\alpha<\lambda)$, $(Q_\beta:\beta<\mu)$ $(\lambda, \mu<k)$ are sequences of k-directed ultrafilters over $\underset{\sim}{A}$ so that for $\alpha<\lambda$, $\beta<\mu$ $P_\alpha\neq Q_\beta$. Suppose $X, Y, Z\subseteq A$, where $1\notin X$, $0\notin Y$, $|X|+|Y|+|Z|<k$, $X$ is directed upward, $Y$ is directed downward, $X<Y$, $Z|<X$,*

$Y \| < Z$, *and* $X \cap (\bigcup_{\beta<\mu} Q_\beta) = \varnothing$, $Y \subseteq \bigcap_{\alpha<\mu} P_\alpha$. *Then there is* $x \in A$ *such that* $X < x < Y$, $x \| Z$, *and* $x \in (\bigcap_{\alpha<\lambda} P_\alpha) \cap (\bigcap_{\beta<\mu} Q^c)$.

To avoid trivialities, we may assume that $1 \in Y$, $0 \in X$ i. e. $X, Y \neq \varnothing$.

P r o o f: Let us define an increasing sequence $(b_\alpha : \alpha < \lambda)$ so that $b_\alpha \in P_\alpha$, $\alpha < \lambda$. By L. 3.6. there is $b_0 \in P_0$ so that $X < b_0 < Y$, and $b_0 \| Z$. Assume $(b_\rho : \rho < \alpha)$ has been defined. Then $\{b_\rho : \rho < \alpha\} < Y$, so by the same lemma there is $b_\alpha \in P_\alpha$ such that $b_\alpha \| Z$, and for all $\rho < \alpha$ $b_\rho < b_\alpha < Y$. Then $\{b_\alpha : \alpha < \lambda\} < Y$ so by the condition $R_k$ there is $b \in A$ such that $b_\alpha < b < Y$ for all $\alpha < \lambda$, and $b \| Z$. Then $X < b < Y$, $b \| Z$, and $b \in \bigcap_{\alpha<\lambda} P_\alpha$.

Dualizing the above procedure, but taking $\{b\}$ instead of $Y$, we can find an element a so that $X < a < b$, $a \in \bigcap_{\beta<\mu} Q^c_\beta$, and $a \| Z$.

By L. 3.7. there is $c \in (\bigcap_{\alpha<\lambda} P_\alpha) \cap (\bigcap_{\alpha<\mu} Q^c_\alpha)$. Let $x = a + bc$. Then $a \leqslant x \leqslant b$, so $X < x < Y$ and $x \| Z$. Since $\bigcap_{\alpha<\lambda} P_\alpha$ is a filter, $\bigcap_{\alpha<\mu} Q^c_\alpha$ an ideal, $b, c \in \bigcap_{\alpha<\lambda} P_\alpha$, and $a, c \in \bigcap_{\alpha<\mu} Q^c_\alpha$, we have also $x \in (\bigcap_{\alpha<\lambda} P_\alpha) \cap (\bigcap_{\alpha<\mu} Q^c_\alpha)$. $\dashv$

T h e o r e m 3.9. *For every cardinal* $\lambda$, $T^*_\lambda$ *is the model completion of* $T_\lambda$.

P r o o f: Obviously

(1) $T_\lambda \subseteq T^*_\lambda$. Now we prove:

(2) For each universal sentence $\varphi \in L(T)$, $T^*_\lambda \vdash \varphi$ implies $T_\lambda \vdash \varphi$.

So assume $T^*_\lambda \vdash \varphi$ and suppose $\sim T_\lambda \vdash \varphi$. Thus, there is a model $\mathfrak{A} \models T_\lambda$ such that $\mathfrak{A} \models \neg \varphi$. Let $\psi$ be a quantifier-free formula such that $\neg \varphi$ is $\exists \bar{x} \psi (x_0 \ldots x_m)$, and assume that $\underline{U}_0, \ldots, \underline{U}_n$ are all ultrafilter's predicates which occur in $\psi$. Thus, $\mathfrak{A}' \models \psi (\overline{a_0, \ldots, a_m})$ for some $a_0, \ldots, a_m \in A$, where $\mathfrak{A}'$ is the reduct of $\mathfrak{A}$ to the language $\overline{L}(T) \cup \{U_0, \ldots, \underline{U}_n\}$. Let $\mathfrak{A}''$ be the submodel of $\mathfrak{A}'$ generated by $a_0, \ldots, a_m$. Then $\overline{\mathfrak{A}}'' = (A'', \overline{V}_0, \ldots, V_n)$ where $A''$ is a Boolean subalgebra of $A$, and $V_i = U_i \cap A''$, $i = 0, \ldots, n$. Hence, $A''$ is finite and has, say, $k$ atoms. Let $B$ be an atomless $BA$, and $c_0, \ldots, c_k \in B$ so that for all $i < k$ $c_i \neq 0$, for $i \neq j$ $c_i c_j = 0$, $\sum_{i<k} c_i = 1$. Hence, there is an embedding $h : A'' \to B$ so that the atoms of $A''$ are sent in $c_0, \ldots, c_{k-1}$. Moreover, we may assume that $h$ is in fact an inclusion. Then there are ultrafilters $P_0, \ldots, P_n$ of $B$ so that $V_i = A'' \cap P_i$, $i = 0, \ldots, n$. Since $\mathfrak{A}'' \models \psi (a_0, \ldots, a_m)$, we also have $\mathfrak{B} \models \psi (a_0, \ldots, a_m)$, where $\mathfrak{B} = (B, P_0, \ldots, P_n)$, i. e. $\mathfrak{B} \models \neg \varphi$. Obviously, $T^*_n \models \varphi$ and $\mathfrak{B} \models T^*_n$, what is contradiction to $\mathfrak{B} \models \neg \varphi$. Hence (2) holds.

For the moment, let us restrict to $\lambda \leqslant \omega$. Thus, $L(T_\lambda)$ is countable, so Blum's criterion, T. 2.12., can be applied. Also, we suppose $\lambda = \omega$ (the case

$\lambda\in\omega$ is handled in a similar way). Therefore, suppose $\mathfrak{B}^*\models T^*_\omega$, $\mathfrak{A}$, $\mathfrak{A}(c)\models T_\omega$, where $\mathfrak{A}(c)$ is a simple extension of $\mathfrak{A}$, $\mathfrak{B}$ a $|A|^+$-saturated model,.

$$\begin{tikzcd} \mathfrak{B}^* & \\ \mathfrak{A} \arrow[u, phantom, "\cup\!|"] \arrow[r, phantom, "\subseteq"] & \mathfrak{A}(c) \arrow[ul, "f"'] \end{tikzcd}$$

$\mathfrak{B}^*\models T^*_\omega$

$\mathfrak{A}, \mathfrak{A}(c)\models T_\omega$

$\mathfrak{B}^*$ is $k^+$-saturated, $|A|=k$.

We have to construct an embedding $f:\mathfrak{A}(c)\to\mathfrak{B}^*$ so that the displayed diagram commutes. Suppose $\mathfrak{A}(c)=(A(c), R_n)_{n\in\omega}=(A(c), S_n, T_n)_{n\in\omega}$, where $c\in S_n$, $c'\in T_n$, $n\in\omega$ ($R_n$ are interpretations of ultrafilter simbols in $\mathfrak{A}(c)$). It might be that there are, in fact, finitely many $R_n$'$s$ with $c\in R_n$, or finitely many $R_n'$s with $c'\in R_n$, but in either case there is no difference in the proof). Further, $\mathfrak{A}=(A, U_n, V_n)_{n\in\omega}$ so that $U_n\subseteq S_n$, $V_n\subseteq T_n$ for all $n\in\omega$, and $\mathfrak{B}^*=(B^*, P_n, Q_n)_{n\in\omega}$, so that $U_n\subseteq P_n$, $V_n\subseteq Q_n$. for all $n\in\omega$. Let $X=\{x\in A: x<c\}$, $Y=\{y\in A: y>c\}$, and $Z=\{z\in A: z\,\|\,c\}$. If we suppose $c\notin A$, what is merely only worth to assume, we have $X<Y$, $0\in X$, $1\in Y$, $Z|<X$, $Y|<Z$, $|X|+|Y|+|Z|<k^+$, $X\cap(\bigcup_{n\in\omega} Q_n)=\varnothing$, $Y\subseteq\bigcap_{n\in\omega} P_n$. Since $\mathfrak{B}^*$ is $k^+$-saturated, $\mathfrak{B}^*$ satisfies $H_k^+$ and $P_n$, $Q_n$ are $k^+$-directed. Hence, the conditions of L. 3.8. are satisfied, so there is $b\in B^*$ such that $X<b<Y$, $b\,\|\,Z$, $b\in(\bigcap_{n\in\omega} P_n)\cap(\bigcap_{n\in\omega} Q_n^c)$. Then the map $f$ defined by $f(cx+c'y)=ax+a'y$ is an embedding from $\mathfrak{A}(c)$ into $\mathfrak{B}^*$, and $f|A=id_A$.

We proved the theorem for $\lambda\leqslant\omega$. Now we consider the case $\lambda>\omega$. Let $\mathfrak{B}$, $\mathfrak{C}\models T^*_\lambda$, $\mathfrak{A}\models T_\lambda$, $\mathfrak{A}\subseteq\mathfrak{B}$, $\mathfrak{C}$, We have to show $(\mathfrak{B}, a)_{a\in A}\equiv(\mathfrak{C}, a)_{a\in A}$. Let $\varphi$ belong to the language $L(T)\cup\{\underline{a}: a\in A\}$. Then $\varphi$ belongs to some finite reduct $L'$ of this language. Let $(\mathfrak{B}', d)_{d\in D}$, $(\mathfrak{C}', d)_{d\in D}$ be reducts of $(\mathfrak{B}, a)_{a\in A}$, $(\mathfrak{C}, a)_{a\in A}$ to the language $L(T)\cup L'\cup\{\underline{d}: d\in D\}$, where $D$ is the Boolean subalgebra of $A$ generated by the constants which names occure in $\varphi$. Assume $\mathfrak{B}\models\varphi$. Then $\mathfrak{B}'\models\varphi$. By the first part of the proof, we have $\mathfrak{B}'\equiv\mathfrak{C}'$, so $\mathfrak{C}\models\varphi$. Hence $(\mathfrak{B}, a)_{a\in A}\equiv(\mathfrak{C}, a)_{a\in A}$. $\dashv$

Now we list some consequences of this theorem.

Corollary 3.10. *Let $\lambda$ be any cardinal. Then:*

1° $T^*_\lambda$ *is submodel complete.*

2° $T^*_\lambda$ *allows eliminiation of quantifiers.*

3° $T^*_\lambda$ *is complete.*

4° *For $\lambda\leqslant\omega$ $\mathfrak{M}(T_\lambda)$, $\mathfrak{M}(T^*_\lambda)$ are Jónsson classes of models. If $\lambda>\omega$ $\mathfrak{M}(T_\lambda)$, $\mathfrak{M}(T^*_\lambda)$ are Jónsson classes of models with $\lambda$-Löwenheim-Skolem property (i. e. with $\lambda^+$-Löwenheim-Skolem property in the sense of* [4], *p.* 84).

5° *If $\lambda\leqslant\omega$, universal-homogeneous models of $T_\lambda$ are exactly saturated models of $T^*_\lambda$. If $\lambda>\omega$ and $k>\lambda$, universal homogeneous models of $T_\lambda$ of cardinality $\geqslant k$ are exactly saturated models of $T^*_\lambda$ of cardinality $\geqslant k$.*

Proof: 1°, 2°, and 3° are consequences of the previous theorem. 4° The case $\lambda \leqslant \omega$ follows from T. 3.9. and T. 2.1. If $\lambda > \omega$, by the proof of T. 2.1., $T_\lambda$, $T_\lambda^*$ still have the amalgamation and the joint embedding properties, thus the assertion holds.

5° if $\lambda \leqslant \omega$ the assertion holds by the previous theorem and T. 2.2. Assume $\lambda > \omega$. Then $\|L(T_\lambda)\| = \lambda$. On the other hand, arguments used in the proof of T. 2.2. can be extended so that they prove: Suppose $\|L(\Gamma)\| = \lambda$ and $\Gamma$ has the model completion $\Gamma^*$. Then a model $\mathfrak{A}$ of $\Gamma$, $|A| \geqslant \lambda^+$, is a full model of $\Gamma$ iff $\mathfrak{A}$ is a saturated model of $\Gamma^*$.

Therefore, the assertion 5° holds. $\dashv$

Remark 3.11. The statement 5° of the previous corollary can be stated in the following form: For any cardinal $\lambda$, universal-homogeneous models of $T_\lambda$ are exactly saturated models of $T_\lambda^*$.

The case $\lambda \leqslant \omega$ of this assertion was proved in C. 3.10, so assume $\lambda > \omega$. We show that, in fact, there are no universal homogeneous models of $T_\lambda$ of cardinality $< 2^\lambda$, neither saturated models of $T_\lambda^*$ of cardinality $< 2^\lambda$. First we prove that there are no saturated models of $T_\lambda^*$ of cardinality $< 2^\lambda$. Let $\mathfrak{A} \models T_\lambda^*$, assume $\mathfrak{A}$ is saturated, and $\mathfrak{A} = (\underset{\sim}{A}, P_\alpha)_{\alpha<\lambda}$. Further, let $\varphi : \lambda \to 2$ and $\Pi(x) = \{P_\alpha^{\varphi(\alpha)}(x) : \alpha < \lambda\}$, where $P^1 \overset{\text{def}}{=} P$, $P^\circ \overset{\text{def}}{=} P^c$. It is easily seen that $\Pi(x)$ is finitely consistent, thus by the saturation of $\mathfrak{A}$ there is $a_\varphi \in \bigcap_{\alpha<\lambda} P_\alpha^{\varphi(\alpha)}$. Also, $\varphi \neq \psi$ implies $a_\varphi \neq a_\psi$, so $|A| \geqslant 2^\lambda$.

Now we prove that there is no even a $\lambda$-universal model of $T_\lambda$ of cardinality less than $2^\lambda$. So assume $\mathfrak{A}$ is a $\lambda$-universal model of $T_\lambda$, $\mathfrak{A} = (A, V_\alpha)_{\alpha<\lambda}$. Further, let $B$ be a free $BA$ with $\lambda$ free generators $a_\beta$, $\beta < \lambda$, and assume $a$ is a free generator of $\underset{\sim}{B}$. Further, let for each $\alpha < \lambda$ $P_\alpha$ be the ultrafilter of $\underset{\sim}{B}$ generated by $\{a\} \cup \{a_\beta : \beta < \alpha\} \cup \{a'_\beta : \beta \geqslant \alpha, a_\beta \neq a\}$, and also $Q_\alpha$ generated by $\{a'\} \cup \{a'_\beta : \beta < \alpha\} \cup \{a_\beta : \beta \geqslant \alpha, a_\beta \neq a\}$. Thus, for every $\varphi : \lambda \to 2$ a model $\mathfrak{B}_\varphi = (B, U_\alpha)_{\alpha<\lambda}$ is defined as follows: If $\varphi(\alpha) = 1$ then $U_\alpha = P_\alpha$, if $\varphi(\alpha) = 0$ then $U_\alpha = Q_\alpha$. In such case $\bigcap_{\alpha<\lambda} U_\alpha^{\varphi(\alpha)} \supseteq (\bigcap_{\alpha<\lambda} P_\alpha) \cap (\bigcap_{\alpha<\lambda} Q_\alpha^c)$, so $a \in \bigcap_{\alpha<\lambda} U_\alpha^{\varphi(\alpha)}$ i. e. $\bigcap_{\alpha<\lambda} U_\alpha^{\varphi(\alpha)} \neq \varnothing$. Since $\mathfrak{A}$ is $\lambda$-universal, $\mathfrak{B}_\varphi$ is embedded into $\mathfrak{A}$, thus $\bigcap_{\alpha<\lambda} V_\alpha^{\varphi(\alpha)} \neq \varnothing$. Hence $|A| \geqslant 2^\lambda$.

Thus, the assertion of the remark is proved.

Theorem 3.12. *Let $\mathfrak{A} = (\underset{\sim}{A}, P_\alpha)_{\alpha<\lambda}$ be a model of $T_\lambda^*$, $k$ an infinite cardinal. Then:*

1° *If $\mathfrak{A}$ is a $k$-saturated model, then $\underset{\sim}{A}$ satisfies $H_\kappa$, and $P_\alpha$ are $k$-directed ultrafilters. If $k > \lambda$ is an infinite cardinal, $\underset{\sim}{A}$ satisfies $H_k$, and for all $\alpha < \lambda$ $P_\alpha$ is $k$-directed, then $\mathfrak{A}$ is a $k$-saturated model.*

2° *If $k = sat(\mathfrak{A})$ and $k > 0$, then $|A| \geqslant k^{\overset{k}{\smile}} + 2^\lambda$.*

Proof: First we prove 2°. So let $k = \mathrm{sat}(\mathfrak{A})$, $\mathfrak{A} \models T_\lambda^*$. Thus, $\mathfrak{A}$ is $k$-saturated. As we have shown in the previous remark, $|A| \geqslant 2^\lambda$. By C. 2.25. $|A| \geqslant k^{\overset{k}{\smile}}$. Hence, $|A| \geqslant k^{\overset{k}{\smile}} + 2^\lambda$.

Now we prove 1°. Assume $|A| \geqslant 2^\lambda$. First suppose that $\mathfrak{A}$ is $k$-saturated. By T. 2.7. follows that $\underset{\sim}{A}$ satisfies $H_k$. Further, let $X \subseteq P$, $|X| < k$. Then $\{\underline{c} < x : x \in X\} \cup \{P_\alpha(c)\}$ is finitely consistent set of formulas, thus by the saturation of $\mathfrak{A}$ there is an element $a \in P_\alpha$, $a < X$. Hence, $P_\alpha$ is $k$-directed.

Now, suppose $\underset{\sim}{A}$ satisfies $H_k$, $P_\alpha' s$ are $k$-directed ultrafilters, and $k > \lambda$. We prove that $\mathfrak{A}$ is $\tilde{k}$-saturated. Let $\Pi(x)$ be a maximal nonoprincipal type over a subset $D$ of $A$, $|D| < k$, and $\underset{\sim}{B}$ the subalgebra of $\underset{\sim}{A}$ generated by $D$. $T_\lambda^*$ allows elimination of quantifiers, thus $\Pi(x)$ is determinated by the set $\Pi'(x)$ of atomic and negatomic formulas contained in $\Pi(x)$. Further, assume $U$ is an ultrafilter's predicate and $t(x)$ a Boolean term over $\underset{\sim}{B}$. Then $t(x) = \underline{a}x + \underline{b}x'$ for some $a, b \in B$. Hence,

$$U(t(x)) \Leftrightarrow (U(a) \wedge U(x)) \vee (U(b) \wedge \neg U(x))$$

Thus, assuming $U^{\mathfrak{A}} = P_\alpha$ for some $\alpha$, we have $U(t(x)) \in \Pi(x)$ iff $a \in P_\alpha$, $x \in P_\alpha$, or $b \in P_\alpha$, $x' \in P_\alpha$. Using the maximality of $U$ we can find, as we did in T. 2.7., sets $X, Y, Z$ so that $\Pi(x) \models\!\!=\!\!| \Gamma(x)$, where $\Gamma(x) = \{\underline{a} < x : a \in X\} \cup \cup \{x < \underline{b} : b \in Y\} \cup \{\underline{c} \parallel x : c \in Z\} \cup \{P_\alpha(x) : \alpha \in I\} \cup \{\underline{P_\alpha}(x') : \alpha \in J\}$, and $I \cap J = \varnothing$, $I \cup J = \bar{\lambda}$. By the consistency of $\Pi(x)$, sets $X, Y, Z$ and ultrafilters $P_\alpha$ satisfy the conditioos in L. 3.8. (observe that $P_\alpha$ for $\alpha \in J$ play the rolle of $Q_\alpha$ in L. 3.8.). Thus, by the same lemma there is $x_0 \in A$ which realizes $\Pi(x)$. $\dashv$

It is interesting to see what these theorems assert for $\lambda \leqslant \omega$.

Example 3.13. First assume $\lambda$ is finite. Then every countable model of $T_\lambda$ is $\omega$-saturated. In fact, all models of $T_\lambda$ are $\omega$-saturated, thus $T_\lambda$ is complete in the logic $\mathfrak{L}_{\infty\omega}$. For infinite $k$, $\mathfrak{A} \models T_\lambda^*$ is $k$-saturated iff $\underset{\sim}{A}$ satisfies $H_k$, and $P_0, \ldots P_n$ are $k$-directed ultrafilters. If $k = \mathrm{sat}(\mathfrak{A})$, then $|A| \geqslant k^{\overset{k}{\smile}}$. In fact, we see that there is no difference between $T = T_0$ and $T_\lambda$. so the most of theorems about $T_\lambda$ can be stated in a similar form, as they were for $T$.

The case $\lambda = \omega$ is probably more interesting. We have:

No model of $T_\omega$ of cardinality $< 2^\omega$ is $\omega$-saturated.

A model $\mathfrak{A} = (\underset{\sim}{A}, P_0, \ldots)$ is $k$-saturated (for $k \geqslant \omega_1$) iff $\underset{\sim}{A}$ satisfies $H_k$ and $P_0, P_1, \ldots$ are $k$-directed ultrafilters.

There is an atomless $BA$ of cardinality $2^\omega$ which is $\omega_1$-saturated and has $2^{2^\omega}$ $\omega_1$-directed ultrafilters. To see that, let $\underset{\sim}{B}$ be a free $BA$ with $2^\omega$ free generetors. Let $D$ be a nonprincipal ultrafilter over $\omega$, and $\underset{\sim}{A} = \underset{\sim}{B}^\omega / D$, the ultrapower of $\underset{\sim}{B}$. Then $\underset{\sim}{A}$ is $\omega_1$-saturated (cf. [3], T. 6.1.1.). On the other hand, there are $2^{2^\omega}$ distinct ultrafilters over $\underset{\sim}{B}$, and for every ultrafilter $U$ over $\underset{\sim}{B}$, $U^\omega / D$ is $\omega_1$-directed ultrafilter over $\underset{\sim}{A}$. Hence, $\underset{\sim}{A}$ has $2^{2^\omega}$ $\omega_1$-directed ultrafilters.

By Shelah's result on the existence of $P$-points in $N^* = \beta N - N$ which is the Stone space of $2^\omega / F$ ($F$ is the filter of cofinite sets over $\omega$), it is consistent with $ZFC$ to assume that there is no expansion of the $BA$ $2^\omega / F$ to an $\omega_1$-saturated model of $T_1$, even if

$2^\omega/F$ is a $\omega_1$-saturated model of $T_0$ (by E. 2.30.). However, if $CH$ is assumed, both Boolean algebras $\underset{\sim}{A}$ and $2^\omega/F$ are $\omega_1$-saturated models and of cardinality $\omega_1$, thus, by the uniqueness of (elementary equivalent) saturated models, $2^\omega/F \cong \underset{\sim}{A}$. Hence, assuming $CH$ we have:

1° There are $2^{2^\omega}$ $\omega_1$-saturated ultrafilters over $\underset{\sim}{2}^\omega/F$, i. e. $2^{2^\omega}$ $P$-points in the space $N^*$ (W. Rudrn, cf. [13]).

2° Given any two sequences $(P_n: n\in\omega)$, $(Q_n: n\in\omega)$ of $\omega_1$-saturated ultrafilters over $2^\omega/F$, there is an automorphism of $2^\omega/F$ so that $f(P_n)=Q_n$. This follows by the uniqueness of saturated models; $(\underset{\sim}{2}^\omega/F, P_0, P_1 \ldots) \equiv (\underset{\sim}{2}^\omega/F, Q_0, Q_1, \ldots)$ and both these models are $\omega_1$-saturated.

There are (at least) two ways of getting $\omega_1$-saturated models of $T_0$ of cardinality $2^\omega$. One is given in E. 2.30., and another one is by means of ultraproducts of countable atomless Boolean algebras. If $CH$ is assumed, isomorphic models are obtained. If $CH$ is not assumed, models obtained by ultraproduct construction have $\omega_1$-saturated ultrafilters (at least $2^\omega$), thus, again by Shelah's result, it cannot be proved in $ZFC$ alone that Boolean algebras obtained by reduced products of Boolean algebras modulo the filter of cofinite subsets of $\omega$ are isomorphic to them (at least this is true for $2^\omega/F$). However, it is interesting to see if models within these two groups are (possibly) isomorphic. For example, it is easily seen that for any two finite Boolean algebras $\underset{\sim}{A}, \underset{\sim}{B}$ holds $\underset{\sim}{A}^\omega/F \cong \underset{\sim}{B}^\omega/F$ (it follows from the fact that $\underset{\sim}{2}^\omega/F = \underset{\sim}{2}^\omega/F x \ldots x \underset{\sim}{2}^\omega/F$).

It would be of some interest to compute sat $(\underset{\sim}{A})$ for so obtained Boolean algebras.[1]

## 4. Applications and remarks.

There will be several applications of the previous consideratins.

4.1. *Model completion of the theory of distributive lattices.*

Let $\Sigma$ be the theory of distributive lattices and $\Sigma^* = \Sigma + \forall\, xy\,(x<y) \Rightarrow \Rightarrow \exists\, z\,(x<z<y) + \forall\, x\, \exists\, y\,(x<y) + \forall\, x\, \exists\, y\,(y<x) + \forall\, xyz\,(x<y \Rightarrow \exists\, u\,(zu=x \wedge \wedge\, z+u=y))$.

We shall prove that $\Sigma^*$ is a model completion of $\Sigma$. It is easy to prove the following assertion:

Proposition 4.1.1. *Let $A$ be an atomless BA, $U$ an ultrafilter and $I$ a maximal ideal over $\underset{\sim}{A}$ so that $\tilde{U} \neq I^c$. Then $(U \cap I, +, \cdot, \leqslant) \models \Sigma^*$.* $\dashv$

Let $\underset{\sim}{M} = (M, +, \cdot, \leqslant)$ be a distributive lattice. As it is well known there is a lattice of sets $F$ such that $\underset{\sim}{M} \cong (F, \cup, \cap, \subseteq)$. Let $\underset{\sim}{B}$ be the field of sets generated $F$ by. Obviously, $B$ is a Boolean algebra. Let $\underset{\sim}{B}(\underset{\sim}{M}) \overset{\text{def}}{=} \underset{\sim}{B}$. We may assume $\underset{\sim}{M} \subseteq \underset{\sim}{B}(\underset{\sim}{M})$. The following property of Boolean algebras is also well known.

Proposition 4.1.2. *Let $\underset{\sim}{M}$ be a distributive lattice, $\underset{\sim}{A}$ a BA, and $f: \underset{\sim}{M} \to \underset{\sim}{A}$ a lattice embedding which preserves end points, if they exist in $\underset{\sim}{M}$. Assume $\underset{\sim}{B}$ is a Boolean subalgebra of $\underset{\sim}{A}$ generated by $f(M)$. Then $\underset{\sim}{B} \cong B(\underset{\sim}{M})$.* $\dashv$

[1] S. Todorčević observed that sat $(2^\omega/F) = \omega_1$ as a consequence of a Hausdorff theorem (cf. [4] Theorem 14.14 and Lemma 14.21).

Proposition 4.1.3. *$\Sigma^*$ is $\omega$-categorical.*

Proof: Let $S$ be a countable model of $\Sigma$ and $\underset{\sim}{A}=B(S)$. Then $\underset{\sim}{A}$ is a countable atomless $BA$. Let $U$ be the filter of $\underset{\sim}{A}$ generated by $S$, and $I$ the ideal of $\underset{\sim}{A}$ also generated by $S$. Then $U$ is an ultrafilter, $I$ is a maximal ideal of $\underset{\sim}{A}$, and $S=U\cap I$. By P. 3.1. $T_2^*$ is $\omega$-categorical, thus, $\Sigma^*$ is $\omega$-categorical. $\dashv$

Corollary 1° *$\Sigma^*$ is complete.*

2° *$\Sigma^*$ is model complete.* $\dashv$

Theorem 4.1.5. *$\Sigma^*$ is a model—completion* of $\Sigma$.

Proof: Obviously

(1) $$\Sigma\subseteq\Sigma^*.$$

Now we prove

(2) Every model of $\Sigma$ is embeddable into a model of $\Sigma^*$.

By compactness theorem it suffices to prove this assertion for finitely generated lattices, that is for finite distributive lattices. So suppose $\underset{\sim}{M}$ is a finite distributive lattice. Let us add two new elements to $M$, the lowest 0, and the greatest 1, i. e. take $N=M\cup\{0, 1\}$, and assume $\underset{\sim}{N}$ is the corresponding lattice. There is a countable atomless $BA$ $\underset{\sim}{A}$ so that $\underset{\sim}{N}\subseteq\underset{\sim}{A}$. Also, there is an ultrafilter $P$ and a maximal ideal $I$ over $\tilde{A}$ so that $\tilde{M}\subseteq\tilde{P}\cap I$. By P. 4.1.1. (2) holds.

Now we prove

(3) For every distributive lattice $\underset{\sim}{M}$, $\Sigma^*+\Delta(\underset{\sim}{M})$ is complete.

By compactness theorem and Löwenheim-Skolem theorem it suffices to prove that for any countable $\underset{\sim}{A}^*, \underset{\sim}{B}^*$ such that $M\subseteq A^*, B^*$, $(A^*, a)_{a\in M}$ is elementary equivalent to $(\underset{\sim}{B}^*, a)_{a\in M}$. So assume $M\subseteq A^*, B^*$, where $\underset{\sim}{A}^*, \underset{\sim}{B}^*$ are countable models of $\Sigma^*$. By P. 4.1.3. we may assume that $A^*=P\cap I$, $B^*=Q\cap J$, where $P$ is an ultrafilter and $I$ a maximal ideal of an atomless, countable $BA$ $\underset{\sim}{C}$. The similar condition is assumed for $Q, J$ and an atomless $BA$ $\underset{\sim}{D}$. Let $\underset{\sim}{A}$ be a Boolean subalgebra of $\underset{\sim}{C}$ generated by $M$, and $\underset{\sim}{B}$ Boolean subalgebra of $\underset{\sim}{D}$ generated also by $M$. We see that $\underset{\sim}{A}\cong B(\underset{\sim}{M})\cong\underset{\sim}{B}$. Now, we prove that there is an isomorphism $f:(\underset{\sim}{A}, P_A, I_A)\to(\underset{\sim}{B}, Q_B, J_B)$, where $P_A=P\cap A$, $I_A=A\cap I$, $Q_B=Q\cap B$, $J_B=J\cap B$, and $f|M=id_M$. Observe that $P_A$ is an ultrafilter over $\underset{\sim}{A}$.

Let $x\in A$. Then $x$ has one of the following forms $(a, b, a_i, b_i\in M)$: $x_0=a$, $x_1=b'$, $x_2=a+b'$, $x_3=\sum_i a_i b_i'$, $x_4=a+\sum_i a_i b_i'$, $x_5=b'+\sum_i a_i b_i'$, $x_6=a+b'+\sum_i a_i b_i'$. Observe that then $x_0, x_4\in P\cap I$, $x_2, x_6\in P\cap I^c$, $x_3\in P^c\cap I$, $x_1, x_5\in P^c\cap I^c$.

The map $f\left(\alpha a+\beta b'+\gamma \sum_i a_i b_i'\right)=\alpha a+\beta \bar{b}+\gamma \sum_i a_i \bar{b}_i$, where $\alpha, \beta, \gamma \in \{0, 1\}$, and $\bar{x}$ is the complement of $x$ in $B$, defines an isomorphism $f:(\underset{\sim}{A}, P_{\underset{\sim}{A}}, I_A) \to (\underset{\sim}{B}, Q_B, J_B)$

$$\begin{array}{ccc} (\underset{\sim}{C}, P, I) & \xrightarrow[f^*]{\sim} & (\underset{\sim}{D}, Q, J) \\ \cup\!\mid & & \cup\!\mid \\ (\underset{\sim}{A}, P_A, I_A) & \xrightarrow[f]{\sim} & (\underset{\sim}{B}, Q_B, J_B) \end{array}$$

By T. 3.9. $T_2^*$ is a model completion of $T_2$, thus, there is an isomorphism $f^*$ so that the displayed diagram commutes. Hence, $(A^*, m)_{m\in M} \equiv (B^*, m)_{m\in M}$.

Remark 4.1.6. The theory $\Sigma'$ of distributive lattices with endpoints (the language $L(\Sigma')$ is $L(\Sigma)$ expanded by $\{0, 1\}$) has as a model-completion: the theory of atomless Boolean algebras, more precizely, the theory of complementary, distributive, dense lattices. Thus, universal homogeneous models for $\Sigma'$ are exactly universal homogeneous Boolean algebras, i. e. saturated atomless Boolean algebras.

4.2. *The filter of cofinite subsets of $\omega$ is $\omega_1$-saturative.*

In [16] the notion of saturative filter is introduced: $A$ filter $D$ over a set $I$ is $\lambda$-saturative iff for every family of models $\mathfrak{E}_i$, $i \in I$, the reduced product $\prod_{i\in I} \mathfrak{E}_i/D$ is $\lambda$-saturated.

The main theorem on saturative filters proved in [16] is:

Theorem 4.2.1. (T. 3.1. *in* [16]) *Assume $D$ is a filter over a set $I$. $D$ is $\lambda$-saturative ($\lambda > \omega$) iff it satisfies the following conditions:*
(1) *$D$ is $\lambda$-good,* (2) *The reduced product $2^I/D$ is $\lambda$-saturated,* (3) *$D$ is incomplete.* $\dashv$

Let $F$ be the filter of cofinite subsets of $\omega$. Obviously, $F$ is incomplete. By E. 2.30., the reduced product $2^\omega/F$ is $\omega_1$-saturated.

Lemma 4.2.2. *$F$ is $\omega_1$-good.*

Proof: We have to show (cf. [3], p. 307) that for every monotonic $f: P_\omega(\omega) \to F$ there is a multiplicative $g: P_\omega(\omega) \to F$ such that $g \leqslant f$ ($P_\omega(\omega)$ denotes the set of finite subsets of $\omega$). Thus, assume $f: P_\omega(\omega) \to F$ is monotonic. For $n \in \omega$, define $G_n = \bigcap\{f(x) : x \subseteq n+1\}$. For $s = \{n_1, \ldots, n_k\}$, let $g(s) = G_{n_1} \cap \cdots \cap G_{n_k}$. Then $g: P_\omega(\omega) \to F$, $g$ is multiplicative, and $g \leqslant f$. $\dashv$

Therefore, we have the following property of the filter $F$ of cofinite subsets of $\omega$.

Proposition 4.2.3. *$F$ is $\omega_1$-saturative.* $\dashv$

For example, let $n = (\{0, 1, \ldots, n-1\}, \leqslant)$. Then, the reduced product $n^\omega/F$ is a saturated Heyting's algebra. By the way, as we have mentioned Heyting's algebras, let us say that every Heyting's algebra which satisfies the separation principle $H_k$ also satisfies the condition $R_k$.

4.3. *Independent families of sets.* The proof of the existence of these families and computing the number of ultrafilters over a set $I$ goes back to $B$.

Pospišil, F. Hausdorff, A. Tarski, Đ. Kurepa and others. However, this approach might be of an interest.

In L. 3.7. it is shown that $k$-directed ultrafilters have the independence property. This lemma can be proved quite easily for finite $\lambda$, $\mu$. In fact, something more holds:

Let $\underset{\sim}{A}$ be any $BA$ and $P_0, \ldots, P_m, Q_0, \ldots, Q_n$ ultrafilters over $A$ so that for $i \neq j$ $P_i \neq Q_j$. Then $(\bigcap_{i \leq m} P_i) \cap (\bigcap_{j \leq n} Q_j^c) \neq \varnothing$.

For the proof of this assertion, let $P$ be any ultrafilter over $A$, and $a_i \in P$, $a_i' \in Q_i$, for $i \leq n$. Let $a = \prod_{i \leq n} a_i$. Then $a \in P$ and $a \in \bigcap_{i \leq n} Q_i^c$. Thus, there is a sequence $b_i$, $i \leq m$, such that $b_i \in P_i$, $b_i \in \bigcap_{j \leq n} Q_j^c$, for $i \leq m$. Let $b = \sum_{i \leq m} b_i$. Then $b \in (\bigcap P_i) \cap (\bigcap Q_j^c)$.

As the consequence of this remark we have at once that the number of ultrafilters over $k$ ($k$ is infinite) is $2^{2^k}$. For that, consider a free Boolean algebra $A$ with $k$ free generators $a_\alpha$, $\alpha < k$. Then every ultrafilter $P$ over $\underset{\sim}{A}$ is uniquelly determinated by a map $\varphi : k \to 2$ and the set $\{a_\alpha^{\varphi(\alpha)} : \alpha < k\} \subseteq P$. Thus, the set $\mathcal{S}$ of all ultrafilters over $A$ has the cardinality $2^k$. Since $|A| = k$, we may assume $|A| = k$. Thus, by the above remark there is a family, that is above $\mathcal{S}$, of independent subsets of $k$ of cardinality $2^k$. Hence, by the usual argument there are $2^{2^k}$ ultrafilters.

In E.3.13. we considered a free $BA$ $\underset{\sim}{B}$ with $2^\omega$ free generators. As we have seen, its ultraprodust $\underset{\sim}{A} = \underset{\sim}{B}^\omega / D$ ($D$ is a nonprincipal ultrafilter over $\omega$) has $2^{2^\omega}$ $\omega_1$ — directed ultrafilters. On the other hand, we may assume $c = |A|$ (here $c = 2^\omega$), since $|A| = c$. Thus, by L.3.7. there is a family $X$ of subsets of $c$ which is $\omega_1$ — independent and of cardinality $2^c$ ($\omega_1$ — independent means: for every $p$: $\omega \to X$, every $\varphi : \omega \to 2$, $\bigcap_{n \in \omega} p_n^{\varphi(n)} \neq \varnothing$, also cf. [17]). From it follows:

Proposition 4.3.2. *There are $2^c$ $\omega_1$ — complete filters over $\omega$.*

Proposition 4.3.3. *Every $\omega_1$ — free Boolean algebra of size $2^c$ is embeddable into the field of subsets of $c$.*

4.4. *Topological interpretations.* By the Stone representation theorem, most of the previous results have topological interpretations. For an object, or notion $S$ of Boolean algebras, let $\hat{S}$ denote its topological dual in the sense of Stone functor. Thus, for a $BA$ $\underset{\sim}{B}$, $\hat{\underset{\sim}{B}}$ denotes the Stone space of $\underset{\sim}{B}$. For example, if $F$ is the filter of cofinite subsets of $\omega$, it is easy to see that $\widehat{2^\omega / F} = N^*$, where $N^* = \beta N - N$, the growth of discrete space on natural numbers.

Since the contravariant functor between the categories of Boolean algebras and Stone spaces is in fact an isomorphism (i.e. a natural equivalence), it is possible translation of most theorems formulated in terms of one category into the terms of another.

In that sense we have the following: If $a$ is a point of $B$ then its topological dual $\hat{a}$ is the clopen set $\{p$: $p$ is an ultrafilter over $\underset{\sim}{B}$ and $a \in p\}$. If

$F$ is a filter over $B$ then $\hat{F}=\{p\in\tilde{B}: F\subseteq p\}$ (a closed subset of $\tilde{B}$). If $I$ is an ideal over $B$ then $\hat{I}=\{p\in\tilde{B}: p\cap I\neq\varnothing\}$, an open subset of $\hat{B}$. If $g$ is a Boolean epimorphism (embedding), then $\hat{g}$ is a continuous embedding (epimorphism). If $p$ is $k$ — directed ultrafilter over $B$, then $p$ is a $P(k)$ — point in $\hat{B}$. In the case $k=\omega_1$, $p$ is a $P$ — point in $\hat{B}$.

For an illustration we will obtain some well known theorems on the space $N^*$. We assume CH. W. Rudin has shown in [13] that every two $P$ — points $p$, $q$ of $N^*$ can be interchanged by an autohomeomorphism of $N^*$. We have remarked already that the dual space of $2^\omega/F$ ($F$ is the Fréchet's filter) is $N^*$, thus, by E.3.13.2$^0$ it follows that not for only two $P$ — points, but for any two countable sequences of $P$ — points $(p_n: n\in\omega)$, $(q_n: n\in\omega)$, where for $i\neq j$, $p_i\neq p_j$, $q_i\neq q_j$, there is an autohomeomorphism $f$ of $N^*$ such that for all $n\in\omega$ $f(p_n)=q_n$.

Also, we obtain very easily (under $CH$) that there are $2^{2^\omega}$ $P$ — points in $N^*$ (W. Rudin). For the proof of this assertion, consider Boolean algebras $2^\omega/F$, and $B^\omega/D$ ($D$ is a nonprincipal ultrafilter over $\omega$), where $B$ is a free $BA$ with $2^\omega$ free generators. Both this algebras are saturated (see E.3.13.) and of cardinality $\omega_1=2^\omega$, so by uniqueness of saturated models $2^\omega/F\cong B^\omega/D$. By E.3.13.1° it follows that $2^\omega/F$ has $2^c$ $\omega_1$ — saturated ultrafilters, i.e. $N^*$ has $2^c$ $P$ — points.

The theorem of I. I. Parovičenko (cf. [13], or Theorem in [17], p. 81) which characterizes the space $N^*$ follows from the $\omega_1$ — saturation of the $BA$ $2^\omega/F$, uniquness of saturated models, and T.2.7.

It is well known that the closed, infinite subsets of $\beta N$ and $N^*$ are of cardinality $2^{2^\omega}$. These are immediate consequnces of P.2.21 and Stone representation theorem.

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