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ON SOME INTERPOLATING INEQUALITIES

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1. In ([1], [2], pp. 189—190 and [3], p. 110) one can find the following inequalities:

(1) $$2\sqrt{n}-2+\frac{1}{\sqrt{n}}<\sum_{k=1}^{n}\frac{1}{\sqrt{k}}<2\sqrt{n}-1 \qquad (n>1)$$

and

(2) $$2\sqrt{n+1}-2<\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \qquad (n=1, 2, \ldots).$$

In [2] it is noted that (2) implies the l. h. s. of (1).

We note that the l. h. s. of (1) implies (2) too, since

$$2\sqrt{n}-2+\frac{1}{\sqrt{n}}<\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \quad (n>1) \Leftrightarrow 2\sqrt{n}-2<\sum_{k=1}^{n-1}\frac{1}{\sqrt{k}} \quad (n>1) \Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{n+1}-2<\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \quad (n\geqslant 1).$$

Since

$$2\left(\sqrt{n+1}-1\right)<2\sqrt{n}-2+\frac{1}{\sqrt{n}} \quad (n=1, 2, \ldots),$$

we note also that the l. h. s. of (1) is sharper than (2).

Our intention in this note is to refine the inequality (2), and furthermore, to provide an infinitely number of new inequalities interpolating the inequality (2) (and also the l. h. inequality of (1) as well). We shall also find infinitely many inequalities which sharpen the r. h. inequality of (1).

2. Proposition 1. For $\alpha<7/16$ and $k\geqslant 1$, and also for $\alpha=7/16$ and $k>1$ there holds

(A) $$\sqrt{k+1-\alpha}+\sqrt{k-\alpha}>2\sqrt{k}.$$

For $\alpha=7/16$ and $k=1$ we have in (A) the equality relation $=$.

Proposition 2. If $1/2 \leqslant \alpha \leqslant 1$ and $k \geqslant 1$ then

$$\sqrt{k+1-\alpha}+\sqrt{k-\alpha}<2\sqrt{k}.$$

Proposition 3. The function defined by

$$\psi_n(\alpha)=\sqrt{n+1-\alpha}-\sqrt{1-\alpha} \qquad (n=1, 2, \ldots)$$

is strictly increasing in the interval $-\infty<\alpha\leqslant 1$.

Proofs of those Propositions result arter a few calculations.

On the basis of the Proposition 1 we get

$$2\left(\sqrt{n+1-\alpha}-\sqrt{1-\alpha}\right)<\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \tag{3}$$

$(\alpha<7/16,\ n=1, 2, \ldots$ and also for $\alpha=7/16,\ n\geqslant 2)$.

For $\alpha=7/16$, $n=1$ we have in (3) the equality relation $=$.

In virtue of the Proposition 2 we have

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}<2\left(\sqrt{n+1-\alpha}-\sqrt{1-\alpha}\right) \qquad (1/2\leqslant\alpha\leqslant 1,\ n=1, 2, \ldots). \tag{4}$$

From (3) for $\alpha=0$ we obtain inequality (2). Due to the Proposition 3, every $0<\alpha<7/16$ by (3) provides a sharper result than (2).

On the other hand, for $\alpha=1/2$, and consequently, by Proposition 3, for $1/2\leqslant\alpha\leqslant 1$, our inequalities (4) are sharper than the r. h. s. of (1), since it is easily to find that for $n>1$ there hold

$$2\left(\sqrt{n+1/2}-\sqrt{1/2}\right)<2\sqrt{n}-1.$$

In virtue of the Proposition 3, for $n>1$ every $1/2\leqslant\alpha\leqslant 9/16$ by (4) gives a sharper result than the r. h. s. of (1).

3. At the end we observe that our inequalities (3) and (4) can be done in the following, simplest but equivalent, forms:

$$2\left(\sqrt{n+9/16}-3/4\right)\leqslant\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \qquad (n=1, 2, \ldots) \tag{3*}$$

with equality only for $n=1$, and

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}<2\left(\sqrt{n+1/2}-\sqrt{1/2}\right) \qquad (n=1, 2, \ldots). \tag{4*}$$

If is easy to find that l. h. s. of (1) is also weaker than (3*) since

$$2\sqrt{n}-2+\frac{1}{\sqrt{n}}<2\left(\sqrt{n+9/16}-3/4\right) \qquad (n>1).$$

A special case of (3) and (4) but better than (1) and (2) was proved in [4].

V. Perić has given a valuable remark. D. Adamović has read this note and has given a number of useful remarks comments and suggestions.

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