PUBLICATIONS DE L'INSTITUT MATHÉMATIQUES
Nouvelle série, tome 25 (40), 1979, pp. 167–169

AN INTUITIONISTIC OMITTING TYPES THEOREM

(Received October 1, 1978)

Zoran Marković

"Intuitionistic" in the title refers to the fact that we are dealing with the logic whose syntax consists in Heyting's predicate calculus and the semantics in so called Kripke models. The approach is classical and consequently no claims are made as to the intuitionistic validity of the results, so the term "intuitionistic" might be somewhat misleading, but it is still used (in the absence of a more precise term) in order to avoid a cumbersome multi-line title. Some explanation is also due of what is meant here by "omitting types theorem". Different equivalent formulations of OTT in classical case might give rise to different theorems in the present setting. However, many of such theorems would not make much sense from model-theoretic point of view. Hopefully, the present one does, and though some improvements might be possible, only those weakening the "locally omitting" condition would be meaningful.

§ 1. Notation

Let $\mathscr{L}$ be a countable, first-order language, T — a consistent (intuitionistic) theory in $\mathscr{L}$, and $\Sigma(x)$ a set of formulas in $\mathscr{L}$ with at most x free. We say that T *locally omits* Σ if for any sentence $\exists x \varphi(x)$ in $\mathscr{L}$, consistent with T, there exists $\sigma(x) \in \Sigma$ such that $\exists x(\varphi(x) \wedge \neg \sigma(x))$ is consistens with T. By a *model of* T we mean a Kripke model $\mathfrak{M} = \langle \mathscr{S}; \mathfrak{A}_s : s \in S\rangle$ (where $\mathscr{S} = \langle S, \leqslant, 0\rangle$ is a p.o. set with the least element 0 and $\mathfrak{A}_s$ are classical structures) such that $0 \Vdash \varphi$ for every $\varphi \in T$. We say that T has a model *omitting* Σ if for each element c of the universe A_0 of the model $\mathfrak{A}_0$ at the base node there is a formula $\sigma(x) \in \Sigma$ such that $0 \Vdash \sigma[c]$. An $\mathscr{L}$-saturated theory T is a consistent, deductively closed set of sentences satisfying:

(1) if $\varphi \vee \psi \in T$ then $\varphi \in T$ or $\psi \in T$

(2) if $\exists x \varphi(x) \in T$ then $\varphi(c) \in T$ for some individual constant $c \in \mathscr{L}$

For the details about Kripke models, and saturated theories the reader should consult [1] or [4]. All other notation is standard, as in e.g. [2].

§ 2.

Theorem. *If T locally omits Σ, then T has a model with a countable universe at each node, omitting Σ.*

Proof: Let C be a countable set of "new" constants, i.e. $\mathcal{L} \cap C = \varnothing$ and let $\mathcal{L}' = \mathcal{L} \cup C$. We will prove the theorem by extending T to an $\mathcal{L}'$-saturated theory T_ω having the property that for every individual constant $c \in \mathcal{L}'$ there is a $\sigma(x) \in \Sigma$ such that $\sigma(c) \notin T_\omega$ (the result we obtain will in fact be slightly stronger; namely for some $\sigma(x) \in \Sigma$, actually $\neg \sigma(c) \in T$, securing thus that $\sigma(c) \notin T'$ for every extension T' of T). Then the canonical model obtained from $S = \{T' : T_\omega \subseteq T', T'$ is $(\mathcal{L}' \cup C')$-saturated for some countable set of "new" constants $C'\}$ is the desired model omitting Σ.

Let $E_0 = \{\exists x \varphi_i(x) : i \in \omega, \varphi_i$ in the language $\mathcal{L}'\}$ and $D_0 = \{\varphi_i \vee \psi_i : i \in \omega, \varphi_i \vee \psi_i$ in the language $\mathcal{L}'\}$ be the lists of all existential and disjunctive sentences, respectively, in the language $\mathcal{L}'$, let $\{c_i : i \in \omega\}$ be an enumeration of C and let $T_0 = T$. Define $T_{n+1}, E_{n+1}, D_{n+1}$, for $n \in \omega$, inductively as follows:

Case 1: $n = 3k$.

Let $\exists x \varphi(x)$ be the first sentence from E_n such that $T_n \vdash \exists x \varphi(x)$ and let c be the first constant from C not occurring in T_n or $\varphi(x)$. Then set $T_{n+1} = T_n \cup \{\varphi(c)\}$, $E_{n+1} = E_n - \{\exists x \varphi(x)\}$, $D_{n+1} = D_n$. It is obvious that T_{n+1} is consistent (if T_n is).

Case 2: $n = 3k+1$.

Let $\varphi \vee \psi$ be the first sentence from D_n such that $T_n \vdash \varphi \vee \psi$. If T_n is consistent with φ, let $T_{n+1} = T_n \cup \{\varphi\}$. If not, ψ has to be consistent with T_n, so put $T_{n+1} = T_n \cup \{\psi\}$. In either case let $D_{n+1} = D_n - \{\varphi \vee \psi\}$, $E_{n+1} = E_n$. Evidently T_{n+1} is consistent (if T_n is).

Case 3: $n = 3k+2$.

Thus far we have constructed $T_n = T \cup \{\varphi_1, \ldots, \varphi_n\}$. Let all the individual constants from C occurring in T_n (i.e. in $\varphi_1, \ldots, \varphi_n$), be among $c_{i_1}, \ldots, c_{i_m}$, c_k $(k \neq i_l,\ l \in \{1, \ldots, m\})$. Let $\varphi(c_{i_1}, \ldots, c_{i_m}, c_k) = \varphi_1 \wedge \cdots \wedge \varphi_n$ and let $\varphi(x) = \exists x_1 \cdots \exists x_m \varphi(x_1, \ldots, x_m, x)$ where $x_1, \ldots, x_m, x$ are individual variables not occurring in $\varphi(c_{i_1}, \ldots, c_{i_m}, c_k)$. Then $\exists x \varphi(x)$ is a sentence in $\mathcal{L}$ consistent with T, so there is a formula $\sigma(x) \in \Sigma$ such that $\exists x (\varphi(x) \wedge \neg \sigma(x))$ is consistent with T. Let $T_{n+1} = T_n \cup \{\neg \sigma(c_k)\}$, $E_{n+1} = E_n$, $D_{n+1} = D_n$. Obviously T_{n+1} is consistent if T_n is.

Finally, let $T_\omega = \bigcup_{n \in \omega} T_n$. To show that T_ω is $\mathcal{L}'$-saturated we have to show the following four facts.:

(1) T_ω is deductively closed, i.e. $T_\omega \vdash \varphi$ implies $\varphi \in T$. We should observe that $T_\omega \vdash \varphi$ iff $T_k \vdash \varphi$ for some $k \in \omega$. But if $T_k \vdash \varphi$ then $T_k \vdash \varphi \vee \varphi$, so for some $3n+1 \geqslant k$, $\varphi \vee \varphi$ will be the first consequence of T_{3n+1} in the list D_{3n+1}, and $T_{3n+2} = T_{3n+1} \cup \{\varphi\}$, so $\varphi \in T_\omega$.

(2) T_ω is consistent, i.e. $T_\omega \not\vdash \wedge$ (where $\wedge$ is the symbol for absurdity). For suppose $T_\omega \vdash \wedge$. Then $T_k \vdash \wedge$ for some k, but this is impossible since the construction was performed in such a way that each T_n is consistent, provided T is consistent and omits Σ.

(3) If $\varphi \vee \psi \in T_\omega$ then $\varphi \in T_\omega$ or $\psi \in T_\omega$. $\varphi \vee \psi \in T_\omega$ means that for some k, $\varphi \vee \psi \in T_k$. But then for some $3n+1$ $\varphi \vee \psi$ is the first consequence of T_{3n+1} in the list D_{3n+1} and consequently either $T_{3n+2} = T_{3n+1} \cup \{\varphi\}$ or $T_{3n+2} = T_{3n+1} \cup \{\psi\}$.

(4) If $\exists x \varphi(x) \in T_\omega$ then $\varphi(c) \in T_\omega$ for some $c \in \mathscr{L}'$. As before, $\exists x \varphi(x) \in T_\omega$ means $\exists x \varphi(x) \in T_k$, for some k; and also for some $3n$, $\exists x \varphi(x)$ is the first consequence of T_{3n} in the list E_{3n}. Then for some $c \in C$, $T_{3n+1} = T_{3n} \cup \{\varphi(c)\}$, so $\varphi(c) \in T_\omega$.

Obviously no $c_k \in C$ can realize Σ, since $T_{3k+3} = T_{3k+2} \cup \{\neg \sigma(c_k)\}$ for some $\sigma(x) \in \Sigma$. If d is an individual constant occurring in T, then $T \vdash \exists x(x=d)$ and $\exists x(x=d) \in E_0$ so for some n, $\exists x(x=d)$ is the first consequence of T_{3n} in the list E_{3n}. For some $c_k \in C$ then $(c_k = d) \in T_\omega$, so d cannot realize Σ.

REFERENCES

[1] P. H. G. Aczél, *Saturated intuitionstic theories*, in Contributions to Matheuiatical logic, ed H. A. Schmidt, K. Schute and H. J. Thiele, North Holland, Amsterdam (1968), pp 1—11.

[2] C. C. Chang and H. J. Keisler, *Model theory*, North Holland, Amsterdam (1973).

[3] M Fitting, *Intuitionistic logic, model theory and forcing*, North Holland, Amsterdam (1969).

[4] C. Smorynski, *Applications of Kripke models*, in Metamathematical investigation of intuitionistic arithmetic and analysis, ed. Troelstra, Springer-Verlag, New-York (1973), pp 324—391.