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ON SOME INTERCALATIONS IN ORDERED SETS*

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0:0. In doctoral Thesis (Kurepa, Đ. 1939) were considered various extensions and intercalations, in ordered sets, among other maximal ones (§ 3:VII, § 4, § 10:4, 11:5). In this connexion there are many works by Krasner, Dokas,... on „Kurepa completions", semi-reals...

In Kurepa [1963] n° 4, n° 6 the following two intercalation conditions for any ordered set $(O, \leqslant)$ were published:

0:1. Condition $C(\alpha)$: Any ordered subset of cardinality $<\aleph_\alpha$ admits in the set an extension in every direction in the sense that for any $X \subseteq O$ such that $|X|<\aleph_\alpha$ and for any ordered set X_0 such that $X_0 \supset X$, $X_0 \setminus X=\{x_0\}$ (irrespective whether $x_0 \in O$ holds or not) there exists a point $p \in O$ such that the identity mapping on X plus the mapping $x_0 \mapsto p(x_0)=p$ be an isomorphism between the ordered sets $X_0=X \cup \{x_0\}$ and $X \cup \{p\}$.

0:2. n-intercalation (or $\equiv I_1(n)$) (n any given cardinal number). For any 3-un ($\equiv$ ordered triplet) (A, B, C) of subsets of O, each of cardinality $<n$, the conditions

$$(0:3) \qquad A<B \quad \text{and} \quad B \,\|\, C$$

imply the existence of a point $p: =p(A, B, C)$ in O such that

$$(0:4) \qquad A<p<B, \quad p \,\|\, C.$$

Consequently, the subsets A, B, C are in the left - half - cone $O(\cdot, p): = \{x \,.\!\cdot\!.\, x \in O,\ x<p\}$ of $(O, \leqslant)$, in the right half cone $O(p, \cdot): =\{x \,.\!\cdot\!.\, x \in O, p<x\}$ and in the complement of the p-cone $O[p]: =\{x \,.\!\cdot\!.\, x \in O,\ x \leqslant p \vee x \geqslant p\}$, respectively.

Since the requested point p in the n-intercalation condition satisfies (0:4) one has necessarily

$$0\,A \cap C = \varnothing = 1\,B \cap C$$

where

$$(0:5) \qquad 0\,A: = \bigcup_{a \in A} (\cdot,\ a],\quad 1\,B = \bigcup_{b \in B} [b,\ \cdot)$$

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and

(0:6) $(\cdot,\ a] = \{x\,.\cdot.\,x \in O,\ x \leqslant a\}$, $[b,\ \cdot) := \{x\,.\cdot.\,x \in O,\ b \leqslant x\}$.

Therefore we can formulate following.

0:7. *Intercalation* 01 (n) (or $I(n)$). For any 3-un (A, B, C) of subsets of $(O, \leqslant)$ each of cardinality $<n$ and such that

(0.8) $$(0A \cup 1B) \cap C = \varnothing$$

there exists a point $p := p(A, B, C)$ of O such that

(0:9) $$A < p < B \quad \text{and} \quad p \,\|\, C.$$

1. Theorem. *For any ordinal number α the conditions $C(\alpha)$, 01 $(\aleph_\alpha)$ are equivalent, i.e. an ordered set $(O, \leqslant)$ satisfies $C(\alpha)$ if and only if $(O, \leqslant)$ satisfies $I(\aleph_\alpha)$.*

1:1. Proof of $\Rightarrow$. Let $x_0 := (A, B, C)$ be any 3-un like in the wording of 01 $(\aleph_\alpha)$; then $X := A \cup B \cup C$ is an ordered subset of power $<\aleph_\alpha$ of $(O, \leqslant)$; let us consider the set $X_0 = X \cup \{x_0\}$; of course, $X_0 \setminus X = \{x_0\}$; let $\leqslant'$ be the order relation in X_0 obtained by extending $(X, \leqslant)$ on setting $A <' x_0 <' B$, $x_0 \,\|_{<'}\, C$. As the set $(O, \leqslant)$ has the property $C(\alpha)$ there is a point $p \in O$ such that the mapping s of X_0 satisfying $s \upharpoonright X = Id_X$ and $sx_0 = p$ be a similarity between $(X_0, \leqslant')$ and $(X \cup \{p\}, \leqslant)$; this means exactly that p is a point in $(O, \leqslant)$ requested by the condition 01 $(\aleph_\alpha)$.

1:2. Proof of $\Leftarrow$. Let $(O, \leqslant)$ be given and let X be any subset of cardinality $<\aleph_\alpha$; let $(X_0, \leqslant')$ be any axtension of $(X, \leqslant)$ obtained by adjoining to X a single point $x_0 \notin X$. Consider:

$$A := \{a\,.\cdot.\,a \in X,\ a <' x_0\},\quad B := \{b\,.\cdot.\,b \in X,\ x_0 <' b\},\quad C := X \setminus (A \cup B).$$

The 3-un (A, B, C) of subsets of $(O, \leqslant)$ satisfies the condition (0:3). As a matter of fact, $A <' x_0 <' B$ thus $A <' B$ and $A < B$ (because $A \cup B \subset O$); further there is no point $c \in C$ such that $c \leqslant a$ for some $a \in A$ because, in opposite case, there would be a $c \leqslant' x_0$ i.e. $c \in A$, A being a left segment of $(X_0, \leqslant')$; but $c \in A$ does not hold because $C \subset X \setminus A$. Analogously one checks easily that $C \cap (B, \cdot) = \varnothing$. Thus (0:3), (0,8) hold; then according to 01 $(\aleph_\alpha)$ there is a point $p \in O$ such that $A < p < B$ and $p \,\|\, C$; then the mapping $s \upharpoonright X_0$ such that $s \upharpoonright X = Id_X$ and $sx_0 = p$ is a requested similarity between $(X_0, \leqslant')$ and the subset $(X \cup \{p\}, \leqslant)$, s extending the identity mapping Id_X. This finishes the proof.

1:3. Remark. If one deals with lattices, then one could assume also that not only $A < B$ but that also A, B be directed upwards and downwards respectively. Exactly so proceeded Negrepontis, *S*. Therefore the intercalation condition [I. Negrepontis, S. 1969] p. 517 in L. 1:6. (cf. also Comfort — Negrepontis [1974] p. 124.) should be compared to the intercalation condition $C(\alpha)$ in Đ. Kurepa [1963] p. 21. We considered the condition still earlier; cf Math. Rev. 52 (1976) # 2888.

2. In connection with the above intercalation condition 01 (n) it is natural to consider also the following.

2:1. Intercalation condition 10 (n). Given an ordered set $(O, \leqslant)$ and a cardinal number n. For any 3-un (A, B, C) of subsets of O, of cardinality $<n$ each, and such that

$$A<B,\ (1\,A\cup 0\,B)\cap C=\varnothing$$

there exists a point $p=p(A, B, C)$ of O such that

$$A<p<B,\quad p\,\|\,C.$$

2:2. One checks readily that the arguments of the section 1 hold on permuting the signs 0, 1. In particular, $C(\alpha)$ and $10\,(\aleph_\alpha)$ are equivalent conditions. Therefore, because of 1 we have the following

2:3. Theorem. For any ordered set $(O, \leqslant)$ and any ordinal number α the intercalation conditions $C(\alpha)$, $01\,(\aleph_\alpha)$, $10\,(\aleph_\alpha)$ are pairwise equivalent.

3. On an intercalation $I_2(n)$. The above n-intercalation: $=I_1(n)$ was considered in connection with ramified sets (: $\equiv$ pseudotrees). For any $(0, \leqslant)$ we may examine the folloving intercalation:

3:1. $I_2(n)$. For any ordered set $(O, \leqslant)$ and any 3-un (A, B, C) of subsets, of cardinality $<n$ each, the conditions

$$A<B,\quad A\cup B\,\|\,C$$

imply the existence of a point $p\in O$ such that

$$A<p<B,\quad p\,\|\,C.$$

In other words (cf. (0:3)) $I_2(n)$ is obtained from the condition $I_1(n)$ on requesting moreover that $A\,\|\,C$. Therefore we have the following

3:2. Lemma. If $(O, \leqslant)$ satisfies the n-intercalation, the more $(O, \leqslant)$ satisfies $I_2(n)$; in other words.

(3:3) $I_1(n) \Rightarrow I_2(n)$ for any cardinal number n.

On the other hand, one has the following

(3:4) Lema. $01\,(n) \Rightarrow I_2(n)$.

P r o o f. At first,

$$A\cup B\,\|\,C \Rightarrow (0\,A\cup 1\,B)\cap C=\varnothing. \tag{3:5}$$

In opposite case, one of the sets $0\,A\cap C$, $1\,B\cap C$ would be $\neq\varnothing$; assume that the first case may occur, i.e. that some point c exists such that $c\in C$ $c\leqslant a$ for some $a\in A$; this would mean that $\neg\, A\,\|\,C$, contrarily to the assumption $(A\cup B)\,\|\,C$. So (3:5) holds. Let us now assume that $(O, \leqslant)$ satisfies $I(n)$; to prove that $(O, \leqslant)$ satisfies $I_2(n)$. As a matter of fact, let (A, B, C) be any 3-un of subsets of $(O, \leqslant)$ of power $<n$ each and such that $A<B$, $(A\cup B)\,\|\,C$; then according to (3:5) the assumptions for application of $01\,(n)$ occur and one has a point $p\in O$ such that $A<p<B$, $p\,\|\,C$, as was requested in the conclusion of $I_2(n)$.

3:6. Remark. If one applies $I_1(n)$ to the ordered set $(O, \geqslant)$ one gets for $(O, \leqslant)$ the following.

Interclation $I^1(n)$. For any 3-un (A, B, C) of subsets of $(O, \leqslant)$ of cardinality $<n$ each, the conditions

$$A<B, \quad A \parallel C$$

imply the existence of a point $p \in O$ such that

$$A<p<B, \quad p \parallel C.$$

3:7. In [Kurepa 1963] section 8 following question was raised: Exhibit an ordered set $(O, \leqslant)$ having the $I(\varkappa_\alpha)$—property but not having the $C(\alpha)$—property. According to 2:3 such a set $(O, \leqslant)$ would satisfy reither $01(\varkappa_\alpha)$ nor $10(\varkappa_\alpha)$.

BIBLIOGRAPHY

[1] Comfort, W. W. — Negrepontis, S. 1974 *The theory of Ultrafilters. Springer Verlag*- Grundlehren der math. Wiss., 211 (1974), X+482.

[2] Dokas, Lambros [1963 a] *Sur certains des ensembles ordonnés munis d'opérations: complétés de Dedekind et de Kurepa des ensembles partiellement ordonnés.* C. R. Acad. Sci. 256 (Paris 1963), 2504—2506, M. R. 26 # 4941.

[3] Kurepa, Đuro [1935] *Ensembles ordonnés et ramifiés,* Thèse. (Paris 1935) VI+140; Publ. Math. Univ. Belgrade 4 (Beograd 1935), 1—138.

[4] Kurepa, Đuro [1963] *On universal ramifeid sets.* Glasnik Mat. fiz. astr. (2) 18 (Zagreb 1963) 17—26.

[5] Mijajlović Žarko, [1979] *Saturated Boolean algebras with ultrafilters,* Publ. Math. 26 (40) (Beograd 1979),

[6] Negrepontis, S. [1969] *The Stone space of the saturated Boolean algebras.* Trans. Amer. Math. Soc. 131 (1969), 515—527.

[7] Dokas, Lambros [1973 b] *Certains complété des ensembles ordonnés munis d'operateurs:* complétés de Krasner. C. R. Ac.Sci. 256 (1963) 3937—3939; M. R. # 3947.

[8] Dokas, L. — Krasner, M. [1976] *Limite projective et inductive et certains complétés d'ensembles ordonnés.* Mathematica Balkanica 6 (Beograd, 1976), 41—46.

[9] Kurepa, Đuro: *A classification of topological space. Z-number of spaces.* Publ. Inst. Mathématique 25 (39), pp 79—89, Beograd (1979).

Erratum corrige

p. 85 lines 12, 14, 15 write ind instead of dim

[10] Kurepa, Đuro: *On some hypotheses concerning trees.* Publications Inst. Mathématique 21 (35) (1977), pp. 99—100.

Erratum corrige

p. 105 in 7:8 Proposition: write P_{15}' instead of the second P_{15}.