PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 26 (40), 1979, pp. 145–156

ON SOLVING A SYSTEM OF BALANCED FUNCTIONAL EQUATIONS ON QUASIGROUPS III

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(Received September 27, 1979)

In the third part of this work, we consider the most general form of a system of balanced functional equations on quasigroups and give formulas of general solution for it.

We ilustrate obtained results by few examples.

For undefined notions and notation see [7] and [1].

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Lemma *Any system* Γ *of general, balanced functional equations on quasigroups is equivalent to the following system: first part* $\Gamma^{=}$ *of the system is composed of equations explicitely expressing reducible operations of* Γ *in terms of its irreducible retracts. Second part* $\overline{\Gamma}$ *of the system consists of equations of some irreducible retracts of operations occuring in* Γ, *with some new operations. Third part* Γ^* *of the system we obtain from* Γ *substituting all reducible operations by their irreducible retracts, making use of equations from* $\Gamma^{=}$ *and substituting any operation each time with a new operation according to equations from* $\overline{\Gamma}$. Γ^* *is irreducible system of general balanced functional equations on quasiqroups.*

Proof: System Γ is consistent. The proof is same as in [7].

Let a solution of Γ (on a set S) be given. We interpret operational symbols from Γ as respective quasigroups from the given solution. The obtained system we denote by $\Gamma[S]$.

Let us introduce, for any quasigroup A occuring in $\Gamma[S]$ more than once, a new operation B replacing A in one occurence. Equality of A and B we add to the new system obtained from $\Gamma[S]$.

After a finitely many steps, we obtain a general system $\Gamma_1^*[S]$ with added equations $\overline{\Gamma}_1[S]$.

Equivalence of $\Gamma_1^*[S]$ and $\Gamma[S]$ is a consequence of $\overline{\Gamma}_1[S]$.

If $\Gamma_1^*[S]$ is reducible, we transform it to an equivalent irreducible system in a following way.

Reducibility implies that there is at least one locally reducible operation B. Repeating the procedure from Lemma of [7 (II)], we have:

(1) $$B(z_1, \ldots, z_k) = B_{L'}\left(z_{s_1}, \ldots, B_{L'\cap L}^{-1} B_L(z_{r_1}, \ldots, z_{r_p}), \ldots, z_{s_q}\right)$$

(1) constitutes system $\Gamma_2^{=}[S] \cdot \Gamma_2^*[S]$ we obtain from $\Gamma_1^*[S]$ substituting

$B_{L'}\left(t_{s_1}, \ldots, B_{L'\cap L}^{-1} B_L(t_{r_1}, \ldots, t_{r_p}), \ldots, t_{s_q}\right)$ for $B(t_1, \ldots, t_k)$.

$\overline{\Gamma}_2[S]$ is $\overline{\Gamma}_1[S]$.

Equivalence of $\Gamma_2^*[S]$ and $\Gamma_1^*[S]$ is a consequence of (1) i.e. $\Gamma_2^{=}[S]$.

(a) If B does not occur in $\overline{\Gamma}_2[S]$, we repeat the procedure with a new locally reducible operation, if any.

(b) If B occurs in $\overline{\Gamma}_2[S]$ and $\Gamma[S]$, then B occurs more than once in $\Gamma[S]$. By $\overline{\Gamma}_2[S]$ introduced some new operations equal to B. Let C be one of them. According to (1):

(2) $$C(z_1, \ldots, z_k) = B_{L'}\left(z_{s_1}, \ldots, B_{L'\cap L}^{-1} B_L(z_{r_1}, \ldots, z_{r_p}), \ldots, z_{s_q}\right)$$

We introduce new operations D, E and F by:

(3) $$D(z_{s_1}, \ldots, x, \ldots, z_{s_q}) = B_{L'}(z_{s_1}, \ldots, x, \ldots, z_{s_q})$$

(4) $$Ex = B_{L'\cap L}\, x$$

(5) $$F(z_{r_1}, \ldots, z_{r_p}) = B_L(z_{r_1}, \ldots, z_{r_p})$$

and we obtained:

(6) $$C(z_1, \ldots, z_k) = D(z_{s_1}, \ldots, E^{-1} F(z_{r_1}, \ldots, z_{r_p}), \ldots, z_{s_q})$$

Substituting $D(t_{s_1}, \ldots, E^{-1} F(t_{r_1}, \ldots, t_{r_p}), \ldots, t_{s_q})$ for $C(t_1, \ldots, t_k)$ in $\Gamma_2^*[S]$, we obtain $\Gamma_3^*[S]$. $\Gamma_3^{=}[S]$ is $\Gamma_2^{=}[S]$ and $\overline{\Gamma}_3[S]$ we obtain from $\overline{\Gamma}_2[S]$ replacing equality of B and C with (3), (4) and (5).

Analogously, we get reduction formulas for all operations equal to B (according to $\overline{\Gamma}_3[S]$).

(c) If B occurs in $\overline{\Gamma}_2[S]$ but not in $\Gamma[S]$, then B is a new operation. From $\overline{\Gamma}_2[S]$ it follows that B is equal to some operation C which occurs both in $\overline{\Gamma}_2[S]$ and $\Gamma[S]$. (2) is then valid for such C.

Choose set $\{y_1, \ldots, y_k\} = \{x_i \mid i \in M\}$ of variables in such a way that $y_i \in f_C(i)$ (i.e. y_i occurs in i-th argument of C) in some equation from $\Gamma_2^*[S]$. Let $z_i = \langle c, y_i \rangle y_i$ for $i = 1, \ldots, k$.

(7) and (8) are L'- and L-consequences of (2), where $L' \cap L = \{r_1\}$.

(7) $$B_{L'}(z_{s_1}, \ldots, B_{L'\cap L}^{-1} B'_{L'\cap L} z_{r_1}, \ldots, z_{s_q}) = C_{L'}(z_{s_1}, \ldots, z_{r_1}, \ldots, z_{s_q})$$

(8) $$B''_{L'\cap L} B_{L'\cap L}^{-1} B_L(z_{r_1}, \ldots, z_{r_p}) = C_L(z_{r_1}, \ldots, z_{r_p})$$

with:

$$B'_{L'\cap L}x = B_L(x, \langle c, y_{r_2}\rangle b_{r_2}, \ldots, \langle c, y_{r_p}\rangle b_{r_p})$$

$$B''_{L'\cap L}x = B_{L'}(\langle c, y_{s_1}\rangle b_{s_1}, \ldots, x, \ldots, \langle c, y_{s_q}\rangle b_{s_q})$$

where $b_j = a_i$ if $y_j = x_i$ for $j = 1, \ldots, k$ and corresponding $i \in M$. $\{r_1\}$-consequence of (2) is:

$$B''_{L'\cap L} B^{-1}_{L'\cap L} B'_{L'\cap L} x = C_{L'\cap L} x \tag{9}$$

From (2), using (7), (8) and (9), we obtain:

$$C(z_1, \ldots, z_k) = C_{L'}\left(z_{s_1}, \ldots, C^{-1}_{L'\cap L} C_L(z_{r_1}, \ldots, z_{r_p}), \ldots, z_{s_q}\right) \tag{10}$$

We see that (10) is completely analogous to (1).

From (10) and equality of B and C, we obtain:

$$B(z_1, \ldots, z_k) = C_{L'}\left(z_{s_1}, \ldots, C^{-1}_{L'\cap L} C_L(z_{r_1}, \ldots, z_{r_p}), \ldots, z_{s_q}\right) \tag{11}$$

Introducing new operations D, E and F with:

$$D(z_{s_1}, \ldots, x, \ldots, z_{s_q}) = C_{L'}(z_{s_1}, \ldots, x, \ldots, z_{s_q}) \tag{12}$$

$$Ex = C_{L'\cap L} x \tag{13}$$

$$F(z_{r_1}, \ldots, z_{r_p}) = C_L(z_{r_1}, \ldots, z_{r_p}) \tag{14}$$

we obtain:

$$B(z_1, \ldots, z_k) = D(z_{s_1}, \ldots, E^{-1} F(z_{r_1}, \ldots, z_{r_p}), \ldots, z_{s_q}) \tag{15}$$

In this case we get $\Gamma_3^*[S]$ from $\Gamma_2^*[S]$ substituting

$$D(t_{s_1}, \ldots, E^{-1} F(t_{r_1}, \ldots, t_{r_p}), \ldots, t_{s_2})$$

for

$$B_{L'}\left(t_{s_1}, \ldots, B^{-1}_{L'\cap L} B_L(t_{r_1}, \ldots, t_{r_p}), \ldots, t_{s_q}\right)$$

and

$$C_{L'}\left(t_{s_1}, \ldots, C^{-1}_{L'\cap L} C_L(t_{r_1}, \ldots, t_{r_p}), \ldots, t_{s_q}\right)$$

for $C(t_1, \ldots, t_k)$. $\Gamma_3^=[S]$ we get from $\overline{\Gamma}_2[S]$ replacing (1) with (10) and $\overline{\Gamma}_3[S]$ from $\overline{\Gamma}_2[S]$ replacing equality of B and C with (12), (13) and (14).

As in a previous case, we obtain reduction formulas for all operations equal to C (according to $\overline{\Gamma}_3[S]$). In both cases (b) and (c), we then repeat the procedure with new localy reducible operation, if any. $\Gamma_2^*[S]$ and $\Gamma_3^*[S]$ are equivalent as follows from (1)—(6) ((2) and (7)—(15)).

In such a way we get sequence $\left(\Gamma_j^=[S], \overline{\Gamma}_j[S], \Gamma_j^*[S]\right)_{j=1,\ldots,l}$ of equivalent systems. The last member of this finite sequence must be the system of desired form.

The procedure and formulas did not depend on S. So $\Gamma_l^=$, $\overline{\Gamma}_l$, Γ_l^* is equivalent to Γ where „retracts" are new functional letters.

Theorem *Let* Γ *be a system of balanced functional equations on quasigroups. Let* Γ^*, $\Gamma^=$, $\overline{\Gamma}$ *be the system obtained from* Γ *as in Lemma, and* $\Gamma^=$ *be system of formulas:*

$$A(x_1, \ldots, x_k) = T_A(A_L, A' A_M, \ldots, A^{(m)} A_N, x_1, \ldots, x_k)$$

for any (reducible) operation A *from* Γ. *The general solution of* Γ *is given by:*

(*)
$$A(x_1, \ldots, x_k) =$$
$$= \langle \cdot, A\rangle^{-1} T_A\left(Q^{\pi_A L}_{A\tilde{L}}, Q^{\pi_A M}_{A\tilde{M}}, \ldots, Q^{\pi_A N}_{A\tilde{M}}, \langle \cdot, A\rangle A_{\{1\}} x_1, \ldots, \langle \cdot, A\rangle A_{\{k\}} x_k\right)$$

where (for all operations B, C *occurring in* Γ^**):*

(1) S *is any nonempty set*

(2) $Q_{B^\sim} = Q_{C^\sim}$ *for* $B \sim C$ *iff diisotopy of* B *and* C *is a consquence of* Γ,

(3) *all permutations* $\pi_B\,(B \sim C)$ *on* $\{i, \ldots k\}$ *are uniquely determined if one of them is given.*

(4) $Q_{B^\sim}$ *is any k-ary loop on* S *(satisfying (8)) iff in every term of* Γ^* *occurs at most one operation from* $B^\sim$.

(5) $R_{B^\sim}$ *is any binary group on* S *(satisfying (8)) iff in some term of* Γ^* *occur at least two operations from* $B^\sim$.

(6) $Q_{B^\sim}$ *is any binary abelian group on* S *iff no exchange of some operations occurring in* $\Gamma^*(B^\sim)$ *with dual operations, transforms* $\Gamma^*(B^\sim)$ *in a system of the first kind.*

(7)
$$\ldots, A_{\{1\}}, \ldots, A_{\{k\}}, \ldots$$

are permutations on S *for which the following equations hold:*

$$\langle \cdot, x_i\rangle_1 = \langle \cdot, x_i\rangle_2$$

for any variable x_i *occuring in any equation of* Γ^*. $\langle \cdot, x_i\rangle_r$ *is a composition* $\langle \cdot, x_i\rangle$ *on the left* $(r=1)$ *or right* $(r=2)$ *term of equation.*

(8)
$$\langle \cdot, B\rangle^{-1} Q^{\pi_B}_{B^\sim}(\langle \cdot, B\rangle B_{\{1\}} x_1, \ldots, \langle \cdot, B\rangle B_{\{k\}} x_k) =$$
$$= \langle \cdot, (C)\rangle^{-1} Q^{\pi_C}_{C^\sim}(\langle \cdot, (C)\rangle C_{\{1\}} x_1, \ldots, \langle \cdot, (C)\rangle C_{\{k\}} x_k)$$

if B *is a new operation and* C *irreducible retract of* (C), *such that equality* $B(x_1, \ldots, x_k) = C(x_1, \ldots, x_k)$ *is from* $\overline{\Gamma}$.

Proof: Follows from Lemma and Theorem of [7 (II)].

Using some known facts from the theory of quasigroups, conditions (4), (5) and (6) (with conditio (8)) can be reformulated in such a way that they are independent of (8).

If loops $Q_{B^\sim}$ and $Q_{C^\sim}$ (from (8)) are different, then from equality of B and C it follows that

$$Q_{B^\sim}(x_1, \ldots, x_k) =$$
$$= \langle \cdot, B\rangle \langle \cdot, (C)\rangle^{-1} Q^{\pi_C \pi_B^{-1}}_{C^\sim}(\langle \cdot, (C)\rangle C_{\{1\}} B^{-1}_{\{1\}} \langle \cdot, B\rangle^{-1} x_1, \ldots,$$
$$\langle \cdot, (C)\rangle C_{\{k\}} B^{-1}_{\{k\}} \langle \cdot, B\rangle^{-1} x_k)$$

so $Q_{B^\sim}$ can be eliminated from formulas of general solution of Γ.

If loops $Q_{B^\sim}$ and $Q_{C^\sim}$ are equal, then $Q_{C^\sim}$ is diisotopic to itself, or equivalvntly $(\alpha_1, \ldots, \alpha_k, \alpha)$ is a arbitrary $\pi_C \pi_B^{-1}$-autotopy

$$\left(\alpha\, Q_{C^\sim}(x_1, \ldots, x_k) = Q_{C^\sim}^{\pi_C \pi_B^{-1}}(\alpha_1 x_1, \ldots, \alpha_k x_k)\right),$$

where:

$$\alpha \langle \cdot, B\rangle = \langle \cdot, (C)\rangle$$

$$\alpha_i \langle \cdot, B\rangle B_{\{i\}} = \langle \cdot, (C)\rangle C_{\{i\}} \text{ for all } i = 1, \ldots, k$$

If $Q_{B^\sim}$ is a loop and $Q_{C^\sim}$ an (abelian) group (dualy) isotopic to it, then there are elements a and b from S, such that:

$$Q_{B^\sim}(x, y) = \langle \cdot, B\rangle \langle \cdot, (C)\rangle^{-1} (\langle \cdot, (C)\rangle \langle \cdot, B\rangle^{-1} x \cdot ab \cdot \langle \cdot, (C)\rangle \langle \cdot, B\rangle^{-1} y)$$

$$(\text{or } Q_{B^\sim}(x, y) = \langle \cdot, B\rangle \langle \cdot, (C)\rangle^{-1} (\langle \cdot, (C)\rangle \langle \cdot, B\rangle^{-1} y \cdot ab \cdot \langle \cdot, (C)\rangle \langle \cdot, B\rangle^{-1} x))$$

where $\cdot$ is $Q_{C^\sim}$ and

$$\langle \cdot, (C)\rangle C_{\{1\}} x = \langle \cdot, (C)\rangle B_{\{1\}} x \cdot a$$

$$\langle \cdot, (C)\rangle C_{\{2\}} x = b \cdot \langle \cdot, (C)\rangle C_{\{1\}} x.$$

If (abelian) groups $Q_{B^\sim}$ and $Q_{C^\sim}$ are equal, then there is an automorphism φ of $\cdot$ (i.e. $Q_{C^\sim}$) an elements a and b such that:

$$\langle \cdot, B\rangle x = a \cdot \varphi \langle \cdot, (C)\rangle x \cdot b$$

$$(I \langle \cdot, B\rangle x = a \cdot \varphi \langle \cdot, (C)\rangle x \cdot b)$$

$$\langle \cdot, (C)\rangle C_{\{1\}} x = \langle \cdot, (C)\rangle B_{\{1\}} x \cdot \varphi^{-1} b$$

$$\langle \cdot, (C)\rangle C_{\{2\}} x = \varphi^{-1} a \cdot \langle \cdot, (C)\rangle B_{\{2\}} x$$

where $Ix = x^{-1}$.

General solutions of general systems of balanced functional equations on quasigroups are in terms of arbitrary loops, (abelian) groups and some permutations while general solutions of systems of balanced functional equations on quasigroups are in terms of loops, (abelian) groups satisfying some conditions (depending on a system) and also permutations, diisotopies of some operations and, in some cases, elements of S.

For any given system, we can transform general solution in the best posible form; in the sense that operations occuring in formulas of general solution are connected by conditions in the simplest form.

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In all examples, for the sake of simplicity, we write A_{123} instead of $A_{\{1, 2, 3\}}$ and analogously for other cases.

Example 1. Let following system Γ be given:

$$A(B(x_1, x_2), x_3, C(x_4, x_5)) = D(x_1, A(x_2, x_3, x_4), x_5)$$

According to procedure from Lemma, we replace Γ by:

$$A(B(x_1, x_2), x_3, C(x_4, x_5)) = D(x_1, F(x_2, x_3, x_4), x_5)$$

$$F(x_2, x_3, x_4) = A(x_2, x_3, x_4)$$

Operations A, D and F are reducible:

$$A(x, y, z) = A_{12}(x, A_2^{-1} A_{23}(y, z))$$

$$D(x, y, z) = D_{13}(D_1^{-1} D_{12}(x, y), z)$$

$$F(x, y, z) = G(x, H^{-1} K(y, z))$$

where $G(x, y) = A_{12}(x, y)$, $H = A_2$ and $K(x, y) = A_{23}(x, y)$.

Γ^* is:

$$A_{12}(B(x_1, x_2), A_2^{-1} A_{23}(x_3, C(x_4, x_5))) =$$

$$= D_{13}(D_1^{-1} D_{12}(x_1, G(x_2, H^{-1} K(x_3, x_4))), x_5)$$

$\Gamma^=$ consists of equalities:

$$A(x, y, z) = A_{12}(x, A_2^{-1} A_{23}(y, z))$$

$$D(x, y, z) = D_{13}(D_1^{-1} D_{12}(x, y), z)$$

and $\overline{\Gamma}$ of equalities:

$$G(x, y) = A_{12}(x, y)$$

$$H = A_2$$

$$K(x, y) = A_{23}(x, y)$$

System Γ^* is of the first kind and all the operations are in the same equivalence class. The general solution of Γ^* is given by:

$$A_{12}(x, y) = A_1 x \cdot A_2 y$$

$$A_{23}(x, y) = A_2 x \cdot A_3 y$$

$$B(x, y) = A_1^{-1}(A_1 B_1 x \cdot A_1 B_2 y)$$

$$C(x, y) = A_3^{-1}(A_3 C_1 x \cdot A_3 C_2 y)$$

$$D_{13}(x, y) = D_1 x \cdot D_3 y$$

$$D_{12}(x, y) = D_1 x \cdot D_2 y$$

$$G(x, y) = D_2^{-1}(D_2 G_1 x \cdot D_2 G_2 y)$$

$$K(x, y) = (D_2 G_2 H^{-1})^{-1}(D_2 G_2 H^{-1} K_1 x \cdot D_2 G_2 H^{-1} K_2 y)$$

where:

$$(16)\qquad \begin{cases} A_1 B_1 = D_1 \\ A_1 B_2 = D_2 G_1 \\ A_2 = D_2 G_2 H^{-1} K_1 \\ A_3 C_1 = D_2 G_2 H^{-1} K_2 \\ A_3 C_2 = D_3 \end{cases}$$

Using Γ we get:

$$G_1 x = A_1 x \cdot \varphi^{-1} b^{-1}$$
$$G_2 x = \varphi^{-1} a^{-1} \cdot A_2 x$$
$$D_2 x = a \cdot \varphi x \cdot b$$
$$K_1 x = A_2 x \cdot \theta^{-1} d^{-1}$$
$$K_2 x = \theta^{-1} c^{-1} \cdot A_3 x$$
$$D_2 G_2 H^{-1} x = c \cdot \theta x \cdot d$$

for some automorphisms φ, θ and some elements a, b, c, d. Using (16) we obtain that $c = a$ and $d = b$ and that φ, θ are trivial automorphisms.

General solution of Γ is given by:

$$A(x, y, z) = A_1 x \cdot A_2 y \cdot A_3 z$$
$$B(x, y) = A_1^{-1}(A_1 B_1 x \cdot a \cdot A_1 y)$$
$$C(x, y) = A_3^{-1}(A_3 x \cdot b \cdot A_3 C_2 y)$$
$$D(x, y, z) = A_1 B_1 x \cdot a \cdot y \cdot b \cdot A_3 C_2 z$$

where $\cdot$ is an arbitrary group and A_1, A_2, A_3, B_1 and C_2 arbitrary permutations of S.

Example 2. Let the following system be given:

$$A(x_1, B(x_2, x_3)) = C(x_2, D(x_3, x_1))$$
$$A(x_1, B(x_2, x_3)) = E(x_3, F(x_1, x_2))$$

System is same as system from Example 2 of [7 (II)]. System from [7 (II)] is in the form of one extended equality while Γ is in the form of two equalities. The general solution of the first system is given by:

$$A(x, y) = A_1 x * A_2 y$$
$$B(x, y) = A_2^{-1}(A_2 B_1 x * A_2 B_2 y)$$
$$C(x, y) = C_2 y * A_2 B_1 x$$
$$D(x, y) = C_2^{-1}(A_1 y * A_2 B_2 x)$$
$$E(x, y) = E_2 y * A_2 B_2 x$$
$$F(x, y) = E_2^{-1}(A_1 x * A_2 B_1 y)$$

where $*$ is an arbitrary abelian group and A_1, A_2, B_1, B_2, C_2 and E_2 arbitrary permutations.

Let us solve the system Γ. We first introduce new operations G and H by:

$$G(x, y) = A(x, y) \quad \text{and} \quad H(x, y) = B(x, y)$$

Γ then becomes general system:

$$A(x_1, B(x_2, x_3)) = C(x_2, D(x_3, x_1))$$

$$G(x_1, H(x_2, x_3)) = E(x_3, F(x_1, x_2))$$

The general solution of this system is given by:

$$A(x, y) = A_1 x \cdot A_2 y \qquad E(x, y) = E_2 y * E_1 x$$

$$B(x, y) = A_2^{-1}(A_2 B_2 y \cdot A_2 B_1 x) \qquad F(x, y) = E_2^{-1}(E_2 F_1 x * E_2 F_2 y)$$

$$C(x, y) = C_2 y \cdot C_1 x \qquad G(x, y) = G_1 x * G_2 y$$

$$D(x, y) = C_2^{-1}(C_2 D_2 y \cdot C_2 D_1 x) \qquad H(x, y) = G_2^{-1}(G_2 H_1 x * G_2 H_2 y)$$

where $\cdot$ and $*$ are groups and $A_1, \ldots, H_2$ permutations such that:

$$A_1 = C_2 D_2 \qquad G_1 = E_2 F_1$$

$$A_2 B_1 = C_1 \qquad G_2 H_1 = E_2 F_2$$

$$A_2 B_2 = C_2 D_1 \qquad G_2 H_2 = E_1$$

Equality of G to A implies isotopy of $\cdot$ to $*$ and, by theorem of Albert, isomorphism as well. There is an element a such that:

$$x * y = a^{-1} \cdot x \cdot y \cdot a$$

$$G_1 x = a \cdot A_1 x$$

$$G_2 x = A_2 x \cdot a^{-1}$$

Using these equalities, from equality of H to B it follows that $\cdot$ is an abelian group and that there are elements p and q such that:

$$a = p \cdot q$$

$$A_2 H_1 x = q \cdot A_2 B_1 x$$

$$A_2 H_2 x = p \cdot A_2 B_2 x$$

and also:

$$G_1 x = pq \cdot A_1 x$$

$$G_2 x = (pq)^{-1} \cdot A_2 x$$

$$x * y = x \cdot y$$

Finally we obtain general solution of Γ:

$$A(x, y) = A_1 x \cdot A_2 y$$
$$B(x, y) = A_2^{-1}(A_2 B_2 y \cdot A_2 B_1 x) = A_2^{-1}(A_2 B_1 x \cdot A_2 B_2 y)$$
$$C(x, y) = C_2 y \cdot A_2 B_1 x$$
$$D(x, y) = C_2^{-1}(A_1 y \cdot A_2 B_2 x)$$
$$E(x, y) = E_2 y \cdot q^{-1} \cdot A_2 B_2 x$$
$$F(x, y) = E_2^{-1}(q \cdot A_1 x \cdot A_2 B_1 y)$$

Let $Px = E_2 x \cdot q^{-1}$. Then $E_2^{-1}(q \cdot x) = P^{-1} x$ and:

$$E(x, y) = Py \cdot A_2 B_2 x$$
$$F(x, y) = P^{-1}(A_1 x \cdot A_2 B_1 y)$$

The form of this general solution is same as solution of system from Example 2 of [7 (II)]. We have similar situation in any such case. So we prefer systems in extended equality form, if posible.

E x a m p l e 3. Let

$$t_k = A(x_1, \ldots, x_{k-1}, A(x_k, \ldots, x_{k+n-1}), x_{k+n}, \ldots, x_{2n-1})$$

for $k = 1, \ldots, n$. System Γ: $t_1 = \cdots = t_n$ is the system of n-ary associativity. n-ary quasigroup A satisfying Γ is called n-group.

In order to solve system Γ, we denote main operation of t_k as $A^{(k)}$ and remaining one as $A^{(n+k)}$. So we obtain the system of general associativy for n-ary quasigroups (example 3 of [7 (II)]). From general solution of this system we see that:

$$A(x_1, \ldots, x_n) = A^{(1)}(x_1, \ldots, x_n) = A_1 x_1 \cdot \ldots \cdot A_n x_n$$

for some group $\cdot$ and permutations $A_1, \ldots, A_n$. It follows that:

$$t_k = A_1 x_1 \cdot \ldots \cdot A_{k-1} x_{k-1} \cdot A_k (A_1 x_k \cdot \ldots \cdot A_n x_{k+n-1}) \cdot A_{k+1} x_{k+n} \cdot \ldots \cdot A_n x_{2n-1}$$

From $t_1 = t_2$ we see that $A_1 A_n = A_n A_1$ which implies that $A_1 x = x \cdot a$ and $A_n x = b \cdot x$.

$t_k = t_n$ $(k = 2, \ldots, n-1)$ then becomes:

$$(16)\qquad \begin{aligned} &A_k (x_k \cdot a \cdot A_2 x_{k+1} \cdot \ldots \cdot A_{n-1} x_{k+n-2} \cdot b \cdot x_{k+n-1}) = \\ &= A_k x_k \cdot \ldots \cdot A_{n-1} x_{n-1} \cdot b \cdot x_n \cdot a \cdot A_2 x_{n+1} \cdot \ldots \cdot A_k x_{k+n-1}. \end{aligned}$$

Choose $x_{k+1}, \ldots, x_{k+n-2}$ in such a way that

$$A_2 x_{k+1} \cdot \ldots \cdot A_{n-1} x_{k+n-2} = a^{-1} \cdot b^{-1}$$

Let $A_{k+1} x_{k+1} \cdot \ldots \cdot A_{n-1} x_{n-1} \cdot b \cdot x_n \cdot a \cdot A_2 x_{n+1} \cdot \ldots \cdot A_{k-1} x_{k+n-2}$ be denoted as $b_k \cdot a_k$ and let $a_2 = a$. Then:

$$A_k (x_k \cdot x_{k+n-1}) = A_k x_k \cdot b_k \cdot a_k \cdot A_k x_{k+n-1}$$

and there are automorphisms φ_k of $\cdot$ such that

$$A_k x = a_k^{-1} \cdot \varphi_k x \cdot b_k^{-1} \quad \text{for} \quad k = 2, \ldots, n-1$$

Let:

$$p_1 = a \cdot a_2^{-1} = e \qquad (e \text{ is the unit of} \cdot)$$

$$p_k = b_k^{-1} \cdot a_{k+1}^{-1} \qquad \text{for } k = 2, \ldots, n-2$$

$$p_{n-1} = b_{n-1}^{-1} \cdot b$$

From (16) we get:

$$\begin{aligned} &\varphi_k p_1 \cdot \varphi_k \varphi_2 x_{k+1} \cdot \varphi_k p_2 \cdot \ldots \cdot \varphi_k p_{n-2} \cdot \varphi_k \varphi_{n-1} x_{k+n-2} \cdot \varphi_k p_{n-1} = \\ (17) \qquad &= p_k \cdot \varphi_{k+1} x_{k+1} \cdot p_{k+1} \cdot \ldots \cdot p_{n-2} \cdot \varphi_{n-1} x_{n-1} \cdot p_{n-1} \cdot x_n \cdot \\ &\cdot p_1 \cdot \varphi_2 x_{n+1} \cdot p_2 \cdot \ldots \cdot p_{k-2} \cdot \varphi_{k-1} x_{k+n-2} \cdot p_{k-1}. \end{aligned}$$

If $x_r = e$ for $r = k+1, \ldots, k+n-2$ we see that:

$$\varphi_k (p_1 \ldots p_{n-1}) = p_k \ldots p_{n-1} p_1 \ldots p_{k-1}.$$

Let $c = p_1 \ldots p_{n-1}$ and $\varphi = \varphi_2$. Then $\varphi c = c$.

From (17) we can prove that:

$$\varphi_r x = (p_1 \ldots p_{r-1})^{-1} \cdot \varphi^{r-1} x \cdot (p_1 \ldots p_{r-1})$$

for $r = 2, \ldots, n-1$ and $\varphi^{n-1} x = cxc^{-1}$.

Finally we get the general solution of Γ:

$$A(x_1, \ldots, x_n) = x_1 \cdot \varphi x_2 \cdot \varphi^2 x_3 \cdot \ldots \cdot \varphi^{n-1} x_n \cdot c$$

where $\cdot$ is an arbitrary group, c any element and φ any automorphism of $\cdot$ for which $\varphi^{n-1} x = cxc^{-1}$ and $\varphi c = c$.

So we obtained representation theorem for n-groups (Hosszú [4]).

Example 4. Let Γ be a balanced functional equation $t_1 = t_2$ which contain only one quasigroup symbol $\cdot$. We denote by $*$ the dual quasigroup of $\cdot$.

We say that balanced equation $t_1 = t_2$ is B-equation*) if:

(a) at least one of t_1, t_2 is not of the form xt, tx for some variable x

(b) there are no variables x and y such that t_1 contains xy as a subterm, and one of xy, yx is a subterm of t_2.

x and y are separated in t if neither xy nor yx occur in t.

$t = t$ is trivial equality. Also, we take as trivial any (equivalent) equality obtained from $t = t$ substituting $t' * t''$ for $t'' \cdot t'$ for some subterms t', t'' of t. $t_1 = t_2$ is almost trivial equality if replacing some of $\cdot$ with $*$ transforms $t_1 = t_2$ into trivial equality.

In [3] Belousov proved:

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*) Belousov calls B-equation *несократимое тождество*. Denes and Keedwell translated this as *irreducible identity*. See [6] p. 69.

Theorem 4.1. *A quasigroup $\cdot$ which satisfies any B-equality Γ is isotopic to a group.*

Recently, Taylor in [5] proved more general:

Theorem 4.2. *Let $t_1 = t_2$ be a balanced equality with the property that t_1 contains a subterm xy and x and y are separated in t_2. Then every quasigroup $\cdot$ which satisfies $t_1 = t_2$ is a group isotope.*

But, there are balanced equalities which do not satisfy condition from Th 4.2 and quasigroup which satisfies one of them still must be a group isotope. $x(yz \cdot u) = (x \cdot yz)u$ is such an equation.

Theorem 4.3. *A quasigroup $\cdot$ satisfying any balanced equality Γ, which is not almost trivial, must be a group isotope.*

Proof: Replace in Γ all occurences of $\cdot$ by new different operations. We get general balanced equality Γ^*.

Suppose that for any operation $A(B)$ from Γ^* $\operatorname{card} A^{\sim} = 2$ i.e. $A \sim B$ iff $A \leftrightarrow B$. Eventualy replacing some of A, B by dual operation, we get equation where A and B have same variables in the first (second) argument.

Repeating the procedure we obtain equation which, by replacing $A(A^*)$ by $\cdot(*)$, becomes a trivial equality. So equality Γ must be almost trivial, contrary to hypothesis.

It follows that $\operatorname{card} A^{\sim} > 2$ for at least one of operations from Γ^*. According to Theorem, $\cdot$ must be a group isotope.

Any equation which satisfies conditions of Th 4.2 cannot be almost trivial and so Th 4.3 is a generalisation of Th 4.2.

Th 4.3 cannot be generalized by finding a wider class of equations (in a language with one binary operation) as follows from:

Theorem 4.4. *There is a quasigroup which satisfies all almost trivial equations and which is not a group isotope.*

Proof: Any commutative quasigroup satisfies any almost trivial equality. Quasigroup $\cdot$ given by:

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	5	3	4
3	3	6	1	2	4	5
4	4	5	2	1	6	3
5	5	3	4	6	1	2
6	6	4	5	3	2	1

is a commutative loop. From $22 \cdot 3 \neq 2 \cdot 23$ it follows that $\cdot$ cannot be isotopic to a group.

On the contrary, from the proof of Th 4.3, we see that following theorem holds.

Theorem 4.5. *Let Γ be any system of balanced functional equations on quasigroups (of various arities).*

If some operation occurs more than once in Γ, replace all occurences of this operation by new, different operations. Let the operation $\cdot$ be replaced by $A_i\,(i \in I)$. If in at least one term from Γ there is more than one operation in some class $A_i^{\sim}$, then $\cdot$ must be a group istope.

Problem 1. Does any group isotope satisfies some equation (in a language with one binary operation) which is not allmost trivial?

If we allow wider language, the answer is trivially yes, because any group isotope satisfies Reidemeister condition which is, by [3], equivalent to:

$$x\,(y \backslash ((z/u)\,v)) = ((x\,(y \backslash z))/u)\,v$$

Problem 2. Find a condition for balanced equalities, such that any quasigroup which satisfies some equation satisfying this condition, must be isotopic to an abelian group.

We give two simple sufficient conditions.

Lemma 4.6. *Let $t_1 = t_2$ be balanced equation which is not almost trivial. If we can find variables x and y such that x occurs in t_1' and t_2'' while y occurs in t_1'' and t_2', where $t_1 = t_1' \cdot t_1''$ and $t_2 = t_2' \cdot t_2''$, then $\cdot$ must be isotopic to an abelian group.*

Lemma 4.7. *Let $t_1 = t_2$ be balanced equality. Replace $\cdot$ in $t_1 = t_2$ by new, different operations. If no exchange of some operations in new equality with dual operations, transforms this equality into equality of the first kind, then $\cdot$ must be isotopic to an abelian group.*

It is interesting to find conditions such that for any equation which does not satisfy this condition there is a quasigroup which satisfies equality and which is not isotopic to an abelian group.

Equality $(x \cdot yz)\,u = (y \cdot xz)\,u$ shows that conditions from L 4.6 and L 4.7 are not sufficiently strong.

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