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ON SOME CLASSES OF LINEAR EQUATIONS, II

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0. Introduction

In [1] we introduced three classes of linear operators acting on a commutative algebra V over **R** or **C**. In order to make this paper self-contained we shall briefly repeat those definitions.

Definition 0.1. *We say that a linear operator on V belongs to the class $H(V)$ if the following condition is satisfied:*

$$L(uv)=uLv \quad \textit{if and only if} \quad u\in\ker L.$$

Definition 0.2. *Suppose that $L\in H(V)$ and suppose that the following condition is satisfied:*

If $x_1, \ldots, x_n$ are $(\ker L)$-linearly independent and if

$$\sum_{k=1}^{n} L^{\nu}(u_k x_k)=\sum_{k=1}^{n} u_k L^{\nu} x_k \qquad (\nu=1, \ldots, n)$$

then $u_1, \ldots, u_n\in\ker L$.

We then say that $L\in K(V)$.

Remark. We say that the vectors $x_1, \ldots, x_n$ are $(\ker L)$-linearly independent if the equality $\sum_{k=1}^{n} u_k x_k=0$, where $u_1, \ldots, u_n\in\ker L$, implies $u_1=\cdots=u_n=0$.

Vectors $x_1, \ldots, x_n$ are $(\ker L)$-linearly dependent if there exist vectors $u_1, \ldots, u_n\in\ker L$, not all zero, such that $\sum_{k=1}^{n} u_k x_k=0$.

D e f i n i t i o n 0.3. *We say that* $L \in D_\alpha(V)$ *if for all* $u, v \in V$ *and for fixed* $\alpha \in \ker L$ *we have*

$$L(uv) = u Lv + v Lu + \alpha\, Lu\, Lv.$$

It was shown in [1] that $D_\alpha(V) \subset K(V) \subset H(V)$.

Certain properties of those classes of operators, and particularly those connected with the equation in x:

$$P(L)\, x = 0$$

where P is a polynomial over **R** or **C**, were examined in [1].

In this paper we shall first be concerned with the general linear n-th order equation

$$\left(\sum_{k=0}^{n} p_k L^{n-k} \right) x = q \qquad (p_0 = 1)$$

where $p_1, \ldots, p_n, q \in V$ are given vectors.

Finally we shall apply the obtained results to some functional equations.

In further text the term *general solution* is taken quite literally, i.e. it is the solution which contains all solutions of the considered equation.

1. First order equations

Let $L \in H(V)$ and consider the equations

(1.1) $$Lx + px = q \qquad (p, q \in V)$$

(1.2) $$Lx + px = 0 \qquad (p \in V).$$

T h e o r e m 1.1. *If* x_1 *is a solution of* (1.2), *then its general solution has the form* $x = ux_1$, *where* $u \in \ker L$ *is arbitrary.*

T h e o r e m 1.2. *If* x_1 *is a solution of* (1.1) *and* x_2 *of* (1.2), *then the general solution of* (1.1) *has the form* $x = x_1 + ux_2$ *where* $u \in \ker L$ *is arbitrary.*

T h e o r e m 1.3. *If* x_1 *and* x_2 *are two distinct solutions of* (1.1), *then its general solution has the form*

(1.3) $$x = x_1 + u(x_2 - x_1),$$

where $u \in \ker L$ *is arbitrary.*

P r o o f. We shall only prove Theorem 1.3. It is easily verified that x, given by (1.3), is a solution of (1.1). Conversely, if x_1 and x_2 are solutions of (1.1), i.e. if

(1.4) $$Lx_1 + px_1 = q, \qquad Lx_2 + px_2 = q,$$

and if x is any other solution of (1.1), then from the equations (1.1) and (1.4) we get

$$L(x-x_1)+p(x-x_1)=0, \qquad L(x_2-x_1)+p(x_2-x_1)=0,$$

and hence

$$\begin{vmatrix} x-x_1 & x_2-x_1 \\ L(x-x_1) & L(x_2-x_1) \end{vmatrix}=0,$$

wherefrom follows (see Theorem 3 from [1])

$$x-x_1=u(x_2-x_1) \qquad (u\in\ker L)$$

or (1.3).

2. We have established the form of the general solution of the equation (1.1). If we actually want to solve that equation, we find that the hypothesis $L\in H(V)$ is too weak.

We therefore suppose that $L\in D_\alpha(V)$ and write the general solution (1.3) in the form

$$ax+b=u \qquad (u\in\ker L). \tag{2.1}$$

Applying L to (2.1) we find

$$(a+\alpha La)Lx+(La)x=-Lb,$$

which together with (1.1) yields

$$\frac{La}{a+\alpha La}=p, \qquad \frac{Lb}{a+\alpha La}=-q,$$

i.e.

$$La=\frac{p}{1-\alpha p}a, \qquad Lb=-q(a+\alpha La). \tag{2.2}$$

Hence, instead of solving the equation (1.1), it is enough to determine particular solutions of simpler equations (2.2) and then to use the formula (2.1).

Examples. (i) If $L=\frac{d}{dt}\in D_0$, this method leads to the standard general solution of the first order linear differential equation:

$$x=e^{-\int p\,dt}\left(C+\int qe^{\int p\,dt}\,dt\right) \qquad (C \text{ arbitrary constant}).$$

(ii) If $L=\Delta$, where $\Delta x(t)=x(t+1)-x(t)$, then $\Delta\in D_1$, and this method leads to the general solution of the first order linear difference equation

$$x(t)=\prod_{\nu=0}^{t-1}(1-p(\nu))\left(\Pi(t)+\sum_{\mu=0}^{b-1}\frac{q(\mu)}{\prod_{\nu=0}^{\mu-1}(1-p(\nu))}\right),$$

where Π is an arbitrary function, periodic with period 1.

3. Equations of order n

We now turn to the nth order equations

$$\left(\sum_{k=0}^{n} p_k L^{n-k}\right) x = q \tag{3.1}$$

and $(p_0 = 1)$,

$$\left(\sum_{k=0}^{n} p_k L^{n-k}\right) x = 0 \tag{3.2}$$

where we suppose that $L \in K(V)$.

Theorem 3.1. *If $x_1, \ldots, x_n$ are* $(\ker L)$*-linearly independent solutions of* (3.2), *then its general solution is $x = \sum_{k=1}^{n} u_k x_k$, where $u_k \in \ker L$ are arbitrary.*

Theorem 3.2. *If $x_1, \ldots, x_n$ are* $(\ker L)$*-linearly independent solutions of* (3.2), *and if y is a solution of* (3.1), *then the general solution of* (3.1) *is $x = y + \sum_{k=1}^{n} u_k x_k$, where $u_k \in \ker L$ are arbitrary.*

Theorem 3.3. *If $x_1, \ldots, x_{n+1}$ are* $(\ker L)$*-linearly independent solutions of* (3.1), *then its general solution is given by*

$$x = x_{n+1} + \sum_{k=1}^{n} u_k (x_k - x_{n+1}), \tag{3.3}$$

where $u_k \in \ker L$ are arbitrary.

Proof. We shall only prove Theorem 3.3. It is easily verified that x given by (3.3) satisfies the equation (3.1). Conversely, first note that if $x_1, \ldots, x_{n+1}$ are $(\ker L)$-linearly independent, then $x_1 - x_{n+1}, \ldots, x_n - x_{n+1}$ are also $(\ker L)$-linearly independent vectors.

Suppose now that $x_1, \ldots, x_{n+1}$ are solutions of (3.1), i.e. that

$$\left(\sum_{k=0}^{n} p_k L^{n-k}\right) x_\nu = q \qquad (\nu = 1, \ldots, n+1). \tag{3.4}$$

If x is any other solution of (3.1), then from (3.1) and (3.4) we get

$$\begin{aligned} &\left(\sum_{k=0}^{n} p_k L^{n-k}\right) (x - x_{n+1}) = 0 \\ &\left(\sum_{k=0}^{n} p_k L^{n-k}\right) (x_\nu - x_{n+1}) = 0 \qquad (\nu = 1, \ldots, n). \end{aligned} \tag{3.5}$$

Eliminating the coefficients $p_1, \ldots, p_n$ from the system (3.5) we find

$$(3.6) \qquad \begin{vmatrix} X & X_1 & \cdots & X_n \\ LX & LX_1 & & LX_n \\ \vdots & & & \\ L^n X & L^n X_1 & & L^n X_n \end{vmatrix} = 0,$$

where $X = x - x_{n+1}$, $X_\nu = x_\nu - x_{n+1}$ $(\nu = 1, \ldots, n)$.

From (3.6) follows that there exist $u_1, \ldots, u_n \in V$ such that

$$(3.7) \qquad L^\nu X = \sum_{k=1}^{n} u_k L^\nu X_k \qquad (\nu = 0, 1, \ldots, n)$$

which implies

$$(3.8) \qquad L^\nu\left(\sum_{k=1}^{n} u_k X_k\right) = \sum_{k=1}^{n} u_k L^\nu X_k \qquad (\nu = 1, \ldots, n).$$

Therefore, since $X_1, \ldots, X_n$ are $(\ker L)$-linearly independent, and $L \in K(V)$, from (3.8) follows that $u_1, \ldots, u_n \in \ker L$. Hence, from (3.7) for $\nu = 0$ we get the required result.

4. Variation of parameters

In this part we shall show how it is possible to obtain a solution of (3.1), provided that the general solution of (3.2) is known. In order to do that we again have to suppose that $L \in D_\alpha(V)$. We first prove a lemma.

Lemma. *Suppose that the equation* $(I + \alpha L)x = 0$ *has the unique solution* $x = 0$. *If the vectors* $x_1, \ldots, x_n$ *are* $(\ker L)$*-linearly independent then the vectors* $x_1 + \alpha L x_1, \ldots, x_n + \alpha L x_n$ *are also* $(\ker L)$*-linearly independent.*

Proof. Let $u_1, \ldots, u_n \in \ker L$ be arbitrary, and put

$$u_1(x_1 + \alpha L x_1) + \cdots + u_n(x_n + \alpha L x_n) = 0,$$

or equivalently

$$(4.1) \qquad u_1 x_1 + \cdots + u_n x_n + \alpha L(u_1 x_1 + \cdots + u_n x_n) = 0.$$

From the hypothesis follows that (4.1) implies

$$u_1 x_1 + \cdots + u_n x_n = 0,$$

and since $x_1, \ldots, x_n$ are $(\ker L)$-linearly independent, we conclude that $u_1 = \cdots = u_n = 0$.

Theorem 4.1. *Suppose that* $L \in D_\alpha(V)$ *and that the general solution of* (3.2) *is known. The problem of determining a particular solution of the n-th order equation* (3.1) *can be reduced to the problem of finding particular solutions of n first order equations of the form* $Lu_k = y_k$ $(k = 1, \ldots, n)$.

Proof. The method we employ is, in fact, the well known variation of parameters method. Indeed, the general solution of the equation (3.2) has the form

$$x=\sum_{k=1}^{n} u_k x_k, \tag{4.2}$$

where $x_1, \ldots, x_n$ are $(\ker L)$-linearly independent solutions of (3.2) and $u_1, \ldots, u_n \in \ker L$ are arbitrary. Suppose that $u_k \notin \ker L$. From (4.2) we find

$$Lx=\sum_{k=1}^{n}(u_k Lx_k+x_k Lu_k+\alpha Lu_k Lx_k). \tag{4.3}$$

If we put

$$\sum_{k=1}^{n}(x_k Lu_k+\alpha Lu_k Lx_k)=0,$$

(4.3) becomes

$$Lx=\sum_{k=1}^{n} u_k Lx_k. \tag{4.4}$$

Again, from (4.4) follows

$$L^2 x=\sum_{k=1}^{n}(u_k L^2 x_k+Lx_k Lu_k+\alpha Lu_k L^2 x_k),$$

i.e.

$$L^2 x=\sum_{k=1}^{n} u_k L^2 x_k,$$

provided that

$$\sum_{k=1}^{n}(Lx_k Lu_k+\alpha Lu_k L^2 x_k)=0$$

Continuing this procedure we arrive at

$$L^{n-1} x=\sum_{k=1}^{n} u_k L^{n-1} x_k,$$

with

$$\sum_{k=1}^{n}(L^{n-2} x_k Lu_k+\alpha Lu_k L^{n-1} x_k)=0.$$

Finally,

$$L^n x=\sum_{k=1}^{n}(u_k L^n x_k+L^{n-1} x_k Lu_k+\alpha Lu_k L^n x_k),$$

and substituting the obtained values for $x, Lx, \ldots, L^n x$ into (3.1) we get

$$\sum_{k=1}^{n}(L^{n-1} x_k Lu_k+\alpha Lu_k L^n x_k)=q. \tag{4.5}$$

The system of equations consisting of (4.5) and the equations

$$\sum_{k=1}^{n} (L^{\nu-1} x_k Lu_k + \alpha Lu_k L^{\nu} x_k) = 0 \qquad (\nu = 1, \dots, n-1) \tag{4.6}$$

is a linear system in $Lu_1, \dots, Lu_n$. Since the vectors $x_1, \dots, x_n$ are $(\ker L)$--linearly independent, in virtue of the Lemma we see that the vectors $x_1 + \alpha Lx_1, \dots, x_n + \alpha Lx_n$ are also $(\ker L)$-linearly independent, and hence using the results from [1] we conclude that the determinant of the considered system is not zero, which means that we can solve the system (4.5)—(4.6) to obtain

$$Lu_1 = y_1, \dots, Lu_n = y_n.$$

This proves the theorem.

5. A functional equation

In [1] we gave some important interpretations of the operator L, such as $\frac{d}{dx}$, $f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$, Δ, $f\Delta$, etc. We now give one more interpretation of those operators.

Let $\omega: \mathbf{R} \to \mathbf{R}$ be a given function, and define the functions $\omega_1, \dots, \omega_n: \mathbf{R} \to \mathbf{R}$ by: $\omega_1(x) = \omega(x)$, $\omega_2(x) = \omega(\omega(x)), \dots$, i.e.

$$\omega_1(x) = \omega(x), \quad \omega_{k+1}(x) = \omega(\omega_k(x)) \qquad (k = 1, 2, \dots, n-1).$$

Consider the functional equation

$$f(\omega_n x) + P_1(x) f(\omega_{n-1} x) + \cdots + P_{n-1}(x) f(\omega_1 x) + P_n(x) f(x) = q(x), \tag{5.1}$$

where $P_1, \dots, P_n, q$ are given functions, f is the unknown function, and $\omega_k x$ denotes $\omega_k(x)$ $(k = 1, \dots, n)$.

If we define the operator L by

$$Lf(x) = f(\omega(x)) - f(x), \tag{5.2}$$

then clearly

$$L^k f(x) = \sum_{\nu=0}^{k} (-1)^{\nu} \binom{k}{\nu} f(\omega_{\nu} x) \qquad (k = 1, \dots, n)$$

where $\omega_0 x = x$.

Hence the equation (5.1) can be written in the form

$$L^n f(x) + p_1(x) L^{n-1} f(x) + \cdots + p_{n-1}(x) Lf(x) + p_n(x) f(x) = q(x), \tag{5.3}$$

where the coefficients $p_1, \dots, p_n$ are uniquely determined by the given coefficients $P_1, \dots, P_n$.

Moreover, it is easily verified that the operator L, defined by (5.2) belongs to the class $D_1(V)$, where V is, in this case, the algebra of all real functions. Thus all the results from Sections 3 and 4 (i.e. Theorems 3.1, 3.2, 3.3, 4.1) regarding the structure of the general solution of (5.3) can be applied to that equation, and hence also to the equation (5.1).

In particular, if $P_1, \ldots, P_n$ are constants (which implies that $p_1, \ldots, p_n$ are also constants) we may apply a result from [1] to get a better picture of the general solution of (5.1).

Namely, in [1] among other things we proved the following theorem.

Suppose that P_n is an n-th degree polynomial over $\mathbf{R}$ *(or* $\mathbf{C}$*) and that $L \in D_\alpha(V)$. If there exist distinct characteristic values $\lambda_1, \ldots, \lambda_n$ of L such that $P_n(\lambda_k)=0$ $(k=1, \ldots, n)$ and if $x_1, \ldots, x_n$ are the corresponding characteristic vectors, then the general solution of the equation $P_n(L)x=0$ is given by $x=\sum_{k=1}^{n} u_k x_k$ where $u_k \in \ker L$ are arbitrary.*

Applying those results to the equation (5.3), where $p_1, \ldots, p_n$ are constants, we see that the problem of determining the general solution of (5.3) is reduced to:

(i) finding one particular solution of (5.3);

(ii) solving the equation $\lambda^n+p_1\lambda^{n-1}+\cdots+p_{n-1}\lambda+p_n=0$, which we suppose to have n distinct roots $\lambda_1, \ldots, \lambda_n$;

(iii) finding the general solution of the equation $f(\omega(x))=f(x)$;

(iv) finding a particular solution for each equation $f(\omega(x))=(1+\lambda_k)f(x)$ $(k=1, \ldots, n)$.

6. An example

Let $\omega(x)=\alpha x$, where α is a positive constant. Then $\omega_k(x)=\alpha^k x$ $(k=1, \ldots, n)$. Any equation

$$f(\alpha^n x)+P_1 f(\alpha^{n-1}x)+\cdots+P_{n-1}f(\alpha x)+P_n f(x)=q(x) \qquad (P_n \in \mathbf{R}) \tag{6.1}$$

can be written in the form

$$(L^n+p_1 L^{n-1}+\cdots+p_{n-1}L+p_n I)f(x)=q(x) \qquad (p_k \in \mathbf{R}),$$

where $Lf(x)=f(\alpha x)-f(x)$, $If(x)=f(x)$.

Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct roots of the polynomial

$$\lambda^n+p_1\lambda^{n-1}+\cdots+p_n. \tag{6.2}$$

The general solution of the equation $Lf(x)=0$, i.e. $f(\alpha x)=f(x)$ is given by

$$f(x)=\Pi(\log_\alpha x),$$

where Π is an arbitrary periodic function with period 1.

Finally $x^{\log_\alpha(1+\lambda)}$ is a particular solution of the equation $Lf(x)=\lambda f(x)$ i.e. $f(\alpha x)=(1+\lambda)f(x)$.

Hence, the general solution of (6.1) is given by

$$f(x)=F(x)+\sum_{k=1}^{n} x^{\log_\alpha(1+\lambda_k)}\,\Pi_k(\log x)$$

where F is a particular solution of (6.1) and $\Pi_1, \ldots, \Pi_n$ are arbitrary periodic functions with period 1.

Moreover, in virtue of Theorem 4.1. we see that knowing $\lambda_1, \ldots, \lambda_n$ we can find $F(x)$.

Remark. If the polinomial (6.2) has multiple roots, it is also possible to arrive at the general solution of (6.1), but in a somewhat more complicated way.

7. A functional equation studied by Poinsot, Pompeiu and Montel

According to Pompeiu [3], Poinsot [2] reduced a geometric problem in conection with the centroid of a triangle to the following functional equation

$$2f(2x)=f(x)+x, \tag{7.1}$$

and, assuming that $f(x)$ is continuous, found the solutions of that equation, namely $f(x)=\frac{x}{3}$.

Pompeiu [3] considered the same functional equation and, using an interesting method, found that (7.1) is also satisfied by

$$f(x)=\frac{C}{x}+\frac{x}{3} \qquad (C \text{ arbitrary constant}) \tag{7.2}$$

claiming that (7.2) is the general solution of (7.1) which is continuous in the neighbourhood of $x=0$.

A year later Pompeiu [4] published that Montel wrote to him about the equation (7.1) and showed that it is satisfied by any function of the form

$$f(x)=\frac{x}{3}+\frac{1}{x}\varphi(\log x), \tag{7.3}$$

where φ is an arbitrary periodic function with period log 2.

We shall apply the methods exposed above to the equation (7.1), which is clearly a special case of (6.1), and will show that (7.3) is indeed the general solution of that equation.

Let $Lf(x)=f(2x)-f(x)$. Then $\ker L=\{\Pi(\log_2 x)\mid \Pi \text{ periodic with period } 1\}$. The equation (7.1) can be written as

$$Lf(x)+\frac{1}{2}f(x)=\frac{1}{2}x,$$

and using the method of Section 2 we see that its general solution is given by

$$a(x)f(x)+b(x)=\Pi(\log_2 x),$$

where Π is an arbitrary periodic function with period 1, and a and b are particular solutions of the system

$$La = \frac{\frac{1}{2}}{1-\frac{1}{2}} a; \qquad Lb = -\frac{1}{2} x (a + La),$$

or, equivalently,

$$a(2x) = 2a(x); \qquad b(2x) - b(x) = -\frac{1}{2} x a(2x). \tag{7.4}$$

Hence, we may take $a(x) = x$, and the second equation in (7.4) becomes

$$b(2x) - b(x) = -x^2.$$

It is evident that a solution of the form $b(x) = Cx^2$ (C = const) should be attempted. We easily find $C = -1/3$.

Therefore the general solution $f(x)$ of (7.1) is given by

$$xf(x) - \frac{1}{3} x^2 = \Pi(\log_2 x)$$

i.e.

$$f(x) = \frac{x}{3} + \frac{1}{x} \Pi(\log_2 x)$$

where Π has the same meaning as above.

Remark. Two functional equations which generalise the equation (7.1), namely

$$mf(mx) = f(x) + \frac{m^2-1}{3} x \qquad (m \in \mathbf{N})$$

$$f(2x) = af(x) + p(x) \qquad (a > 0)$$

were also considered in [4]. Clearly, both are special cases of (6.1) and can be solved by the methods given in this paper.

REFERENCES

[1] J. D. Kečkić, *On some classes of linear equations*, Publ. Inst. Math. (Beograd) **24** (38) (1978), 89—97.

[2] L. Poinsot, *Eléments de statique*, Paris 1811, pp. 175—178.

[3] D. Pompeiu, *Une équation fonctionnelle à propos du centre de gravité de l'aire d'un triangle*, Bull. Math. Phys. Ecole. Polytechn. Bucarest **8** (1937). 3—5.

[4] D. Pompeiu, *Sur l'équation fonctionelle de Poinsot et autres équations analogues*, Bull. Math. Phys. Ecole. Polytechn. Bucarest **9** (1938), 54—56.

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