ON THE ESSENTIAL FLATS OF GEOMETRIC LATTICES

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Abstract. Crapo [1] has defined the essential flats, but here they are defined in another way. It is proved that the complements of essential flats are the essential flats of the dual matroid and that the essential flats themselves constitute a lattice.

Matroid M on a finite set (carrier) S is the ordered pair (S, f), where f is the function which maps 2^S to 2^S and which satisfies the following axioms for each $X, Y \subseteq S$ and for each $a, b \in S$:

(1) $$X \subseteq fX$$

(2) $$X \subseteq Y \Rightarrow fX \subseteq fY$$

(3) $$ffX = fX$$

(4) $$b \in f(X \cup a) \backslash fX \Rightarrow a \in f(X \cup b)$$

A subset X of the set S is a *flat* (shortly — F) of the matroid M iff $fX = X$.

The flats of a matroid constitute a lattice (so-called *geometric* lattice) ordered by inclusion, in which the infimum and the supremum are defined by:

(5) $$X \wedge Y = X \cap Y$$

(6) $$X \vee Y = f(X \cup Y)$$

We also use another definition of matroid:

Matroid M on a carrier S is the ordered pair (S, r), where r is the function which maps 2^S to the set of non-negative integers and satisfies the following conditions for each $X, Y \subseteq S$:

(7) $$0 \leqslant rX \leqslant |X|$$

(8) $$X \subseteq Y \Rightarrow rX \leqslant rY$$

(9) $$r(X \cup Y) + r(X \cap Y) \leqslant rX + rY$$

Besides, for each $X \subseteq S$ and for each $y \in S$ is satisfied:

(10) $$rX \leqslant r(X \cup y) \leqslant rX + 1$$

The connection between the functions r and f is given by:

(11) $y \in fX \Leftrightarrow r(X \cup y) = rX$

The number rX is the *rank* of the set X. The *bases* of a matroid M are the minimal subsets among those subsets Y of S which satisfy $rY = rS$.

It is known [2] that the complements according to the carrier S of the bases of a matroid M are the bases of another, *dual*, matroid M^* with the same carrier S.

All objects of the matroid M are denoted by adding (*) to the denotations of the corresponding objects of M (e. g. r^*, f^*, F^*).

The connection between the functions r and r^* is given by:

(12) $r^*(S \backslash X) = |S| - rS - |X| + rX$

for each $X \subseteq S$.

A flat X of a matroid M with the carrier S is an *essential flat* (shortly — *EF*) if it is satisfied

(13) $r(X \backslash y) = rX$

for each $y \in X$.

The *null* and the *unit* of a matroid are the flats of rank 0 and rS respectively. The null is always an *EF*, by (7) and (8).

On the complements of essential flats

In what follows we take that the set X is a subset of the carrier S of a matroid M.

Theorem 1.

(14) X is $F \Rightarrow (X$ is $EF \Leftrightarrow S \backslash X$ is $F^*)$

We divide the proof into two parts:

(15)   ① X is $EF \Rightarrow S \backslash X$ is F^*

(16)   ② $(X$ is $F \wedge S \backslash X$ is $F^*) \Rightarrow X$ is EF

We first prove three lemmas:

Lemma 1.1.

(17) $r(X \backslash y) = rX \Leftrightarrow r(X \backslash y) \neq rX - 1$

for each $X \subseteq S$ and $y \in S$

Proof: If we replace the set X in (10) by $X \backslash y$, then we obtain:

$r(X \backslash y) = rX \vee r(X \backslash y) = rX - 1$

because the rank is an integer-valued function.

Lemma 1.2.

(18) $y \in X \Rightarrow (r(X \backslash y) = rX - 1 \Leftrightarrow y \in f^*(S \backslash X))$

for each $X \subseteq S$ and $y \in S$.

P r o o f: Using (11) and (12) we find:

$$y \in f^*(S \backslash X) \Leftrightarrow r^*(S \backslash (X \backslash y)) = r^*(S \backslash X) \Leftrightarrow$$

$$\Leftrightarrow |S| - rS - |X \backslash y| + r(X \backslash y) = |S| - rS - |X| + rX \Leftrightarrow$$

$$\Leftrightarrow |X| - (|X| - 1) = rX - r(X \backslash y) \Leftrightarrow r(X \backslash y) = rX - 1$$

L e m m a 1.3.

(19) $$y \in X \Rightarrow (r(X \backslash y) = rX \Leftrightarrow y \notin f^*(S \backslash X))$$

for each $X \subseteq S$ and $y \in S$.

P r o o f: The direct consequence of (17) and (18).

The proof of the assertion ①:

Using (13), (12) and (11) we obtain:

$$X \text{ is } EF \Rightarrow r(X \backslash y) = rX \Rightarrow (\forall y \in S)(y \in X \Rightarrow r(X \backslash y) = rX) \Rightarrow$$

$$\Rightarrow (\forall y \in S)(y \in X \Rightarrow y \notin f^*(S \backslash X)) \Rightarrow (\forall y \in S)(y \in f^*(S \backslash X) \Rightarrow y \notin X) \Rightarrow$$

$$\Rightarrow f^*(S \backslash X) \subseteq S \backslash X \Rightarrow f^*(S \backslash X) = S \backslash X \Rightarrow S \backslash X \text{ is } F^*$$

N o t e. All the implications, except the first one, could be reversed.

The proof of the assertion ②:

Let us assume that the assertion is wrong. Then there exists a subset $X \subseteq S$, which satisfies

$$X \text{ is } F \wedge S \backslash X \text{ is } F^* \wedge \neg (X \text{ is } EF)$$

We use that

$$S \backslash X \text{ is } F^* \Leftrightarrow f^*(S \backslash X) = S \backslash X$$

Then we have (by (13), (17) and (18)):

$$X \text{ is } F \wedge \neg (X \text{ is } EF) \Rightarrow X \text{ is } F \wedge \neg (X \text{ is } F \wedge (\forall y)(r(X \backslash y) = rX)) \Rightarrow$$

$$= rX)) \Rightarrow X \text{ is } F \wedge (\neg (X \text{ is } F) \vee \neg (\forall y)(r(X \backslash y) = rX)) \Rightarrow$$

$$\Rightarrow X \text{ is } F \wedge (\exists y)\neg (r(X \backslash y) = rX) \Rightarrow X \text{ is } F \wedge (\exists y)(r(X \backslash y) = rX - 1) \Rightarrow$$

$$\Rightarrow (\exists y)(r(X \backslash y) = rX - 1) \Rightarrow (\exists y)(y \in X \wedge r(X \backslash y) = rX - 1) \Rightarrow$$

$$\Rightarrow (\exists y)(y \in X \wedge y \in f^*(S \backslash X)) \Rightarrow (\exists y)(y \in X \wedge y \in S \backslash X) \Rightarrow (\exists y)(y \in \emptyset)$$

A contradiction.

This completes the proof of the Theorem 1.

The consequences of the Theorem 1.

(20) $$((1)) \quad X \subseteq S \Rightarrow (X \text{ is } EF \Leftrightarrow S \backslash X \text{ is } EF^*)$$

P r o o f: We first prove

(21) $$(X \subseteq S \wedge X \text{ is } EF) \Rightarrow S \backslash X \text{ is } EF^*$$

By the use of (15) and the definition of EF we have

(22) $$(X \subseteq S \wedge X \text{ is } EF) \Rightarrow (X \text{ is } F \wedge S \backslash X \text{ is } F^*)$$

Noticing that $(F^*)^* = F$ (because $(M^*)^* = M$) and that $S\backslash(S\backslash X) = X$ we prove

(23) $(S\backslash X$ is $F^* \wedge X$ is $F) \Rightarrow S\backslash X$ is EF^*

similarly as (16).

The assertion (21) follows from (22) and (23).

In a very similar (dual) way we prove

(24) $(X \subseteq S \wedge S\backslash X$ is $EF^*) \Rightarrow X$ is EF

The assertion (20) follows from (21) and (24).

((2)) The set of matroids whose geometric lattices have the empty set and the carrier as the only essential flats, is selfdual.

((3)) The connection (12) enables us to determine the ranks of all essential flats of the dual matroid. According to [1], the essential flats together with their ranks uniquely determine the geometric lattice (and the matroid). This implies that the Theorem 1. gives an algorithm for construction of the geometric lattice of the dual matroid.

On the lattice of essential flats of a geometric lattice

T h e o r e m 2. *Essential flats of a geometric lattice constitute a lattice ordered by inclusion.*

L e m a 2.1.

(25) $(X$ is $EF \wedge Y$ is $EF) \Rightarrow f(X \cup Y)$ is EF

Before proving this lemma we introduce a special designation and prove five lemmas concerning it;

(26) $q(Y, X) \overset{\text{def}}{=} |Y| - |X| - rY + rX$

for each two subsets X and Y of the carrier.

L e m m a 2.2.

(27) $Y \supseteq X \Rightarrow q(Y, X) \geqslant 0$

P r o o f: Adding some elements of the set $Y\backslash X$ to the set X, the rank of the set X is increased by $rY - rX$. Due to $Y \supseteq X$ and (10) we have

$$|Y| - |X| = |Y\backslash X| \geqslant rY - rX$$

which proves the lemma.

L e m m a 2.3.

(28) $q(Z, X) = q(Z, Y) + q(Y, X)$

P r o o f: The direct consequence of (26).

L e m m a 2.4.

(29) $q(X \cup Y, X) + q(X \cup Y, Y) \geqslant q(X, X \cap Y) + q(Y, X \cap Y)$

P r o o f. The required non equality follows from the next two non-equalities:

1) $q(X \cup Y, X) + q(X \cup Y, Y) \geqslant q(X \cup Y, X \cap Y)$

2) $q(X, X \cap Y) + q(Y, X \cap Y) \leqslant q(X \cup Y, X \cap Y)$

The proofs of these are easily deduced from (9) and (26).

L e m m a 2.5.

(30) $$q(X \cup Y, X) = 0 \Rightarrow q(Y, X \cap Y) = 0$$

Due to (28) we have

(31) $$q(X \cup Y, X) + q(X, X \cap Y) = q(X \cup Y, Y) + q(Y, X \cap Y)$$

We first put $q(X \cup Y, X) = 0$ in (29) and (31) and, after that, replace $q(X \cup Y, Y)$ in (31), using (29). So we obtain

$$q(X, X \cap Y) \geqslant q(X, X \cap Y) + 2 q(Y, X \cap Y)$$

Hence by (27) follows (30).

The sense of the designation $q(X, Y)$ is explained by the following lemma:

L e m m a 2.6.

(32) $$Z \text{ is } EF \Leftrightarrow Z \text{ is } F \wedge (\forall X)((X \subseteq Z \wedge X \neq Z \wedge X \text{ is } F) \Rightarrow q(Z, X) \neq 0)$$

P r o o f: (($\Leftarrow$)) Let the right side hold and Y be a flat which satisfies $Y \subseteq Z$ and $rY = rZ - 1$. Then, according to (27), it also holds $q(Z, Y) > 0$, that is

(33) $$|Z| - |Y| > 1$$

If $x \in Z$, then there exists a flat Y which satisfies

$$Y \subseteq Z \backslash x \wedge r Y = rZ - 1$$

(Furthermore, using (10) and (11) we can prove that each chain of a geometric lattice between the null and an arbitrary flat Z contains exactly one flat of each rank between 0 and rZ.)

According to (10) (or (17)) we have

$$rZ \geqslant r(Z \backslash x) \geqslant rZ - 1 = rY$$

By (33) the set $(Z \backslash x) \backslash Y$ is non-empty. By (8) and (11) this implies that $r(Z \backslash x) > rY$, that is $r(Z \backslash x) = rZ$. This means that Z is EF.

(($\Rightarrow$))   Let Z be an EF. Then by (13)

$$(Y \text{ is } F \wedge Y \subseteq Z \wedge rY = rZ - 1) \Rightarrow |Z| - |Y| > 1,$$

that is, $q(Z, Y) > 0$.

If holds

$$W \text{ is } F \wedge W \subseteq Z \wedge rW < rZ - 1,$$

then in an arbitrary chain of the geometric lattice between the flats W and Z there exists a flat T of rank $rZ - 1$. Due to (28) $q(Z, W) = q(Z, T) + q(T, W)$. As $q(Z, T) > 0$, and, by (27), $q(T, W) \geqslant 0$, so is $q(Z, W) > 0$.

The proof of Lemma 2.1.:

Let A and B be essential flats.

If $A \subseteq B$ (similarly if $B \subseteq A$), then $f(A \cup B) = f(B) = B$, and so $f(A \cup B)$ is EF.

If none of the sets A and B is a subset of the other, then let us suppose that $f(A \cup B)$ is not an EF. Then by (32) we obtain

(34)　　$(\exists X)\,(X \subseteq f(A \cup B) \land X \neq f(A \cup B) \land X$ is $F \land q\,(f(A \cup B), X) = 0$

We differentiate two cases and give a special proof for each:

$$\text{I. } A \subseteq X \qquad\qquad \text{II. } A \nsubseteq X$$

I. We first prove that $A \subseteq X$ implies that $B \nsubseteq X$. The remainder of the proof is quite similar as in the second case.

Let us suppose that $A \subseteq X$ and $B \subseteq X$.

Then we have $A \cup B \subseteq X \subseteq f(A \cup B)$ and, by (8), also $r(A \cup B) \leqslant rX \leqslant$ $\leqslant r(f(A \cup B))$. As from (11) follows

$$r(f(A \cup B)) = r(A \cup B), \quad \text{so is } \quad rX = r(f(A \cup B)).$$

However, the set $f(A \cup B) \backslash X$ is non-empty by (34) and so by (11) and (8) we have that $rX < r(f(A \cup B))$. A contradiction.

II. Using (34) and (1) we prove that $X \cup A \subseteq f(A \cup B)$ By (28) we have $q(f(A \cup B), X) = q(f(A \cup B), X \cup A) + q(X \cup A, X)$. As $q(f(A \cup B), X) = 0$, so by (27) is also $q(X \cup A, X) = 0$. Finally, (30) implies that $q(A, A \cap X) = 0$. As $A \cap X$ is flat, $A \cap X \subseteq A$ and $A \cap X \neq A$, so by (32) the set A is not an EF.

A contradiction.

The proof of Theorem 2. By (25) we have that if A and B are essential flats, then $f(A \cup B)$ is EF, too. As $f(A \cup B)$ is the smallest flat which contains A and B, so it is certainly the smallest EF which contains A and B. Thus the supremum of two subsets A and B in the (ordered by inclusion) set of essential flats (of a geometric lattice) is also defined by $A \lor B = f(A \cup B)$. As the set of essential flats contains the null of the geometric lattice, which is a subset of all essential flats, so the set of essential flats is a lattice, ordered by inclusion.

We denote the lattice of essential flats of a geometric lattice L by $\underline{E_L}$.

Some traits of the lattice E_L

The intersection of two essential flats is not always an EF. For example, if $\langle a, b, c \rangle$ and $\langle a, d, e \rangle$ are flats of rank 2, and $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle d \rangle$, $\langle e \rangle$ are flats of rank 1 of a geometric lattice, then $\langle a, b, c \rangle$ and $\langle a, d, e \rangle$ are essential flats, but $\langle a \rangle$ is not.

In what follows we take that S is the carrier of the matroid with the geometric lattice L.

Theorem 3. *The infimum of two essential flats A and B in the lattice E_L is defined by*

$$A \land B = S \backslash f^*(S \backslash (A \cap B))$$

Proof: The sets $S \backslash A$ and $S \backslash B$ are EF^* by (20). Then by (25) the set $X = f^*(S \backslash (A \cap B))$ is also EF^*. Hence by (20) the set $S \backslash X$ is EF. As X is the smallest EF^* which contains $S \backslash A$ and $S \backslash B$, so $S \backslash X$ is the largest EF which is contained in A and B.

T h e o r e m 4. *The lattice E_L has a unit.*

P r o o f: The null Z of the lattice L^* is the smallest EF^*. So the set $S \setminus Z$ is the largest EF of the lattice L.

C o n s e q u e n c e: *The unit (the carrier) of the lattice L is EF iff the null of the lattice L^* is the empty set.*

We conclude that the lattice E_L is not always a sublattice L. The supremums and nullas are common for these lattices, but it is not always the case concerning their infimums and units.

The lattices E_L and E_{L^*} are mutually inverted. Namely, the mapping $i : X \to S \setminus X$, where $X \in E_L$ is, by (20), a bijection of E_L onto E_{L^*}. Furthermore, according to the proof of Theorem 3., it holds

$$i(X \wedge Y) = i(X) \vee * i(Y),$$

and dually also holds

$$i(X \vee Y) = i(X) \wedge * i(Y)$$

Generally speaking, the lattice E_L is "less regular" than the lattice L, because the lengths of different maximal chains of the lattice E_L, between some two essential flats, are not always equal.

Finally, we cite a theorem which is proved with the help of essential flats:

T h e o r e m 5. *There are n non-isomorphic matroids and $n-2$ non-isomorphic simple matroids of rank $n-1$ on an n-element carrier.*

P r o o f: (Sketch) It is primarily proved that the geometric lattices of matroids of rank $n-1$ on an n-element carrier have exactly one non-empty EF and that the matroids are completely determined by the rank of that EF. The value of that rank is between 0 and $n-1$, therefore there exists n possibilities. The geometric lattices of simple matroids have no non-empty flats of rank 0 and 1.

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