

ON NONLINEAR SUPERPOSITION, III

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(Received February 16, 1979)

0. The principle of nonlinear superposition for nonlinear differential equations (ordinary or partial) is contained in the following: If u_1, \dots, u_n are solutions of a differential equation (E), then $F(u_1, \dots, u_n)$ is also a solution of that equation. The function F is then called a *connecting function* for (E).

Paper [1] which introduced the above definition initiated a certain amount of research in that direction.

In this note we give some remarks which are relevant to nonlinear superposition.

1. D. Pompeiu [2] noted the following:

(i) If y_1 and y_2 are solutions of the equation

$$(1.1) \quad y' = p(x)(1+y) \log(1+y),$$

then

$$(1.2) \quad y_1 + y_2 + y_1 y_2 \quad \text{and} \quad \frac{y_2 - y_1}{1 + y_2}$$

are also solutions of that equation;

(ii) If y_1 and y_2 are solutions of the equation

$$(1.3) \quad f'(y)y' = p(x)f(y),$$

then

$$(1.4) \quad f^{-1}(f(y_1) + f(y_2))$$

is also a solution of that equation;

(iii) For every differential equation

$$(1.5) \quad y' = p(x)B(y),$$

there exists a connecting function F , i.e. if y_1 and y_2 are solutions of (1.5), then there exists a function F such that $F(y_1, y_2)$ is also a solution of (1.5).

Hence, the first trace of the nonlinear superposition principle is to be found in [2], though Pompeiu did not develop his ideas.

2. In [3] we proposed the following conjecture:

Suppose that u_1 and u_2 are arbitrary solutions of a nonlinear equation $N(u)=0$. A connecting function $F(u_1, u_2)$ for the equation $N(u)=0$ can be found if and only if the equation $N(u)=0$ can be transformed into a linear homogeneous equation $L(v)=0$ by means of a substitution of the form $u=g(v)$.

In that case, if n denotes the order of the equation $N(u)=0$, we find that:

(i) For $n=1$, F is a connecting function for $N(u)=0$ if and only if

$$(2.1) \quad F(u_1, u_2) = f^{-1} \left(f(u_2) \varphi \left(\frac{f(u_1)}{f(u_2)} \right) \right),$$

where φ is an arbitrary function, and $f=g^{-1}$.

(ii) For $n \geq 2$, F is a connecting function for $N(u)=0$ if and only if

$$(2.2) \quad F(u_1, u_2) = f^{-1} (C_1 f(u_1) + C_2 f(u_2)),$$

where C_1 and C_2 are arbitrary constants, and $f=g^{-1}$.

3. Pompeiu's examples are in accordance with the above. Indeed, the equations (1.1), (1.3), (1.5) can be reduced to the linear equation $v' = p(x)v$ by means of the transformations $y=e^v-1$, $y=f^{-1}(v)$, and $y=A^{-1}(\log v)$, where $A'(t)B(t)=1$, respectively.

The connecting functions (1.2) for (1.1) are obtained from (2.1) for $f(t)=\log(1+t)$, and $\varphi(t)=1 \pm t$; the function (1.4) from (2.1) for $\varphi(t)=1+t$, while it is easily seen that there exists an infinity of connecting functions for the equation (1.5).

4. It is interesting to note the relation of the connecting functions (2.1) and (2.2) to the general solutions of first and second order equations.

If $N(u)=0$ is an ordinary differential equation, then the ratio $f(u_1)/f(u_2)$ is constant. Therefore, we conclude that the knowledge of one solution of $N(u)=0$ implies the knowledge of its general solution.

If $N(u)=0$ is a partial differential equation, and if u_1 and u_2 are not linearly dependent, then (2.1) is the general solution of that equation.

For second order equations, formula (2.2) gives the general solution if the equation is an ordinary differential equation, and a particular solution containing two arbitrary constants if the equation is a partial differential equation.

5. A. Haimovici ([4], [5]) also used nonlinear superposition for solving the nonlinear boundary value problem

$$(5.1) \quad u_{xx} + cu_{tt} + a(x)u_x + b(t)u_t = 0 \quad (c = \text{const})$$

$$(5.2) \quad u_x(0, t) = \alpha(t)V(u(0, t)), \quad u_x(1, t) = \beta(t)V(u(1, t))$$

where V is given. He introduced the transformation $J(u) = \int dz/V(z)$ which transforms (5.1)—(5.2) into

$$(5.3) \quad J_{xx} + cJ_{tt} + A(J)J_x^2 + cA(J)J_t^2 + B(x)J_x + D(t)J_t = 0$$

$$(5.4) \quad J_x(0, t) = \alpha(t), \quad J_x(1, t) = \beta(t).$$

The equation (5.3) is a special case of the equation

$$G'(V)(aV_{xx} + bV_{xy} + cV_{yy} + dV_x + eV_y) + G''(V)(aV_x^2 + bV_xV_y + cV_y^2) + gG(V) = 0$$

which, according to our hypothesis, is the most general equation for which nonlinear superposition can be developed.

6. Consider the linear equations with functional coefficients

$$(6.1) \quad ay'' + by' + cy = 0$$

$$(6.2) \quad AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + GU = 0.$$

If we look a connecting function for (6.1) in the form $F(u, v)$, we find that

$$(6.3) \quad F(u, v) = uf\left(\frac{v}{u}\right)$$

and

$$(6.4) \quad (u'v - v'u)^2 f''\left(\frac{v}{u}\right) = 0.$$

Hence, if u and v are linearly independent, then $f''\left(\frac{v}{u}\right) = 0$, i.e. $f\left(\frac{v}{u}\right) = C + D\frac{v}{u}$, where C and D are arbitrary constants, which gives the connecting function $F(u, v) = Cu + Dv$, i.e. the linear superposition principle.

If u and v are linearly dependent, then $u'v - v'u = 0$, and f is arbitrary. But then $v = Ku$ ($K = \text{const}$), and the connecting function (6.3) becomes $F(u, v) = Cu$ (C arbitrary constant).

The same procedure applied to the partial differential equation (6.2) yields

$$(6.5) \quad F(u, v) = uf\left(\frac{v}{u}\right),$$

$$(A(vu_x - uv_x)^2 + B(vu_x - uv_x)(vu_y - uv_y) + C(vu_y - uv_y)^2) f''\left(\frac{v}{u}\right) = 0.$$

Comparing (6.4) with (6.3) we see that W , defined by

$$W^2 = A(vu_x - uv_x)^2 + B(vu_x - uv_x)(vu_y - uv_y) + C(vu_y - uv_y)^2,$$

is a formal generalisation of the Wronskian $w \equiv u'v - v'u$, particularly since $v = Cu$ ($C = \text{const}$) implies that $W = 0$. However, $w = 0$ if and only if $v = Cu$, whereas W can vanish when $v \neq Cu$. Indeed, $W = 0$ implies that $v = \varphi(\omega)u$, where φ is an arbitrary function and $A\omega_x^2 + B\omega_x\omega_y + C\omega_y^2 = 0$.

Hence, if $W \neq 0$, from (6.3) we get $f''\left(\frac{v}{u}\right) = 0$ which again leads to the linear superposition principle $F(u, v) = Cu + Dv$ (C, D constants). On the other hand, if $W = 0$ then f is arbitrary, and the connecting function F becomes

$$F(u, v) = uf\left(\frac{v}{u}\right) = uf(\varphi(\omega)),$$

i.e.

$$F(u, v) = u\psi(\omega) \quad (\psi \text{ arbitrary function})$$

and we arrive at a particular solution of (6.2) containing one arbitrary function.

Remark. We shall return to the concept of generalised Wronskians in an other paper, and will examine their connection with the solutions of partial differential equations. A generalisation of the Wronskian can also be found in [6].

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