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A FIXED POINT THEOREM IN MENGER SPACES

Olga Hadžić

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Abstract: In [1], [2], [3], [4], [10], [11] and [15] some fixed point theorems in probabilistic metric and random normed space $(S, \mathcal{F}, t)$ are proved where $t = \min$. We shall prove in this paper a fixed point theorem in complete Menger spaces $(S, \mathcal{F}, t)$ where t is a continuous T-norm such that the family $\{T_n(x)\}_{n \in N}$ is equicontinuous at the point $x = 1$, where;

$$T_n(x) = \underbrace{t(t(t(\cdots t(t}_{n\text{-times}}(x, x), x), \ldots), x), \quad x \in [0, 1], \ n \in N.$$

An example of such T-norm is given. A similar result for matric spaces is obtained in [8]. We shall also give, using the method from [3], a short proof of Theorem 1 from [4].

A probabilistic metric space $(S, \mathcal{F})$ i formed by a nonempty set S together with a mapping $\mathcal{F}$ which assigns to each $(x, y) \in S \times S$ a distribution function $F_{x,y}$ such that the following conditions are satisfied;

(F 1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$.

(F 2) For every $(x, y) \in S \times S : F_{x,y}(0) = 0$.

(F 3) For every $(x, y) \in S \times S : F_{x,y} = F_{y,x}$.

(F 4) If $F_{x,y}(r) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(r+s) = 1$.

By a Menger space $(S, \mathcal{F}, t)$ we mean [13] a probabilistic metric space $(S, \mathcal{F})$ with (F 4) replaced by the condition;

(F 4)′ For every $(x, y, z) \in S \times S \times S$ and every $r, s > 0$;

$$F_{x,z}(r+s) \geqslant t(F_{x,y}(r), F_{y,z}(s))$$

where t is a T norm [13].

The (ε, λ)-topology is introduced by the family $\{U_v(\varepsilon, \lambda)\}_{\substack{v \in S \\ \varepsilon > 0 \\ \lambda \in (0,1)}}$ where $U_v(\varepsilon, \lambda) = \{u \mid F_{u,v}(\varepsilon) > 1 - \lambda\}$ and this topology is metrisable if $\sup_{x<1} t(x, x) = 1$ [13].

Further we shall use the following Theorem ([9] p.p.. 45): *If t is a continuous T-norm and $I=[0,1]$ then:*

$$I\times I=(\bigcup_{k\in K} J_k xJ_k)\cup C(\bigcup_{k\in K} J_k xJ_k)$$

where the set K is at most denumerable, for every $k\in K$ is J_k an open interval, $J_k\cap J_r=\emptyset$ for $k\neq r$ and the restriction $t|J_k xJ_k$ is an Archimedean semigroup i.e. $t(x, x)<x$ for every $x\in J_k$.

A semigroup $t:[A, B]\times[A, B]\to[A, B]$ is *strict* if for every $x\in(A, B)$ the inequality $x_1<x_2$ $(x_1, x_2\in[A, B])$ implies that $t(x, x_1)<t(x, x_2)$.

In the next Theorem we shall denote $t|J_k xJ_k$ by t_k $(k\in K)$.

Theorem *Let $(S, \mathcal{F}, t)$ be a complete Menger space with continuous T-norm t such that the family $\{T_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$ and for every $k\in K$ the semigroup t_k is strict. Further, let the mapping $H:S\to S$ be such that for every $r_i>0$ $(i=1, 2, 3, 4, 5)$ and every $u, v\in S$:*

$$F_{Hu, Hv}\left(\sum_{i=1}^{5} r_i\right)\geqslant t\,(t\,(t\left(F_{u, Hv}\frac{t_5}{a}\right),\ F_{v, Hu}\left(\frac{r_4}{b}\right)),\ F_{v, Hv}\left(\frac{r_3}{c}\right)),$$
$$F_{u, Hu}\left(\frac{r_2}{d}\right)),\ F_{u, v}\left(\frac{r_1}{e}\right))$$

where $a, b, c, d, e\in R^+$ and $a+b+c+d+e<1$. Then there exists a unique fixed point of the mapping H.

Proof; Let us prove that there exists a sequence $\{a_n\}_{n\in N}\subset(0, 1)$ such that $\lim_{n\to\infty} a_n=1$ and that the family $\{d_n\}_{n\in N}$ of pseudometrics defines the (ε, λ)-topology, where;

$$d_n(x, y)=\sup\{t\,|\,F_{x, y}(t)\leqslant a_n\}\quad (n\in N;\ x, y\in S) \tag{1}$$

Suppose that, for every $k\in K$, $J_k=(c_k, b_k)$. If the set K is empty i.e. $t=\min$ we can take for $\{a_n\}_{n\in N}$ an arbitrary sequence from the interval $(0, 1)$ such that $\lim_{n\to\infty} a_n=1$ and from [3] it follows that for every $n\in N$ is d_n pseudometric. Further, it is obviuous that the family $\{d_n\}_{n\in N}$ defines the (ε, λ)-topology. If the set K is finite say $K=\{1, 2, \ldots, m\}$ then $b_m<1$ or $b_m=1$. In the case that $b_m<1$ the restriction $t|[b_m, 1]\times[b_m, 1]=\min$ and we can take for $\{a_n\}_{n\in N}$ an arbitrary sequence from the interval $(b_m, 1)$ such that $\lim_{n\to\infty} a_n=1$. We shall show that the relation $b_m=1$ leads to a contradiction with the asssuption that the family $\{T_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$. Namely, if $b_m=1$ then the restriction $t|[c_m, b_m]\times[c_m, b_m]$ is an Archimedean semigroup such that for every $x\in J_m=(c_m, 1)$: $\lim_{n\to\infty} T_n(x)=c_m$ which fact was proved by Cho--Hsin Ling ([9]). So if $\lambda\in(0, 1)$ is such that $1-\lambda>c_m$ then there is no $\eta(\lambda)\in(0, 1)$ such that;

$$x>\eta(\lambda)\ \Rightarrow\ T_n(x)>1-\lambda$$

which is a contradiction. Suppose now that the set K is infinite. If there exists $b\in(0, 1)$ such that $b_m\leqslant b$ for every $m\in K$ then we can take, as before, for $\{a_n\}_{n\in N}\subset(0, 1)$ an arbitrary sequence such that $a_n>b$, for every $n\in N$,

and $\lim_{n\to\infty} a_n = 1$. In the case that there is no $b \in (0, 1)$ such that $b_m \leqslant b$ for every $m \in K$ then there exists a subsequence $\{c_{m(n)}\}_{n\in N}$ such that $\lim_{n\to\infty} c_{m(n)} = 1$. Then by definition $a_n = c_{m(n)}$ for every $n \in N$. In [7] is proved that the family $\{d_n\}_{n\in N}$ from (1) is also a family of pseudometrics which defines the (ε, λ)-topology, using the method from [3] and the facts that $t(a_n, a_n) = a_n$, for every $n \in N$, and that for every $k \in K$ the semigrup t_k is strict. Now, we shall prove that for every $n \in N$ and every $u, v \in S$ the following relation holds;

$$d_n(Hu, Hv) \leqslant e \cdot d_n(u, v) + d \cdot d_n(u, Hu) + c \cdot d_n(v, Hv) + b \cdot d_n(v, Hu) + \\ + a \cdot d_n(u, Hv)$$

Suppose, on the contrary, that there exist $n \in N$ and $u, v \in S$ such that;

$$d_n(Hu, Hv) > e \cdot d_n(u, v) + d \cdot d_n(u, Hu) + c \cdot d_n(v, Hv) + b \cdot d_n(v, Hu) + \\ + a \cdot d_n(u, Hv).$$

Then there exist $r_1, r_2, r_3, r_4, r_5 \in R^+$ such that;

$$e \cdot d_n(u, v) < r_1; \quad d \cdot d_n(u, Hu) < r_2; \quad c \cdot d_n(v, Hv) < r_3; \quad b \cdot d_n(v, Hu < r_4;$$

$a \cdot d_n(u, Hv) < r_5$ and $d_n(Hu, Hv) > \sum_{i=1}^{5} r_1$. Using the definition of pseudometrics d_n from this it follows that;

$$F_{u,v}\left(\frac{r_1}{e}\right) > a_n: \quad F_{u,Hu}\left(\frac{r_2}{d}\right) > a_n: \quad F_{v,Hv}\left(\frac{r_3}{c}\right) > a_n: \tag{2}$$

$$F_{v,Hu}\left(\frac{r_4}{b}\right) > a_n: \quad F_{u,Hv}\left(\frac{r_5}{a}\right) > a_n: \quad F_{Hu,Hv}\left(\sum_{i=1}^{5} r_i\right) \leqslant a_n. \tag{3}$$

Let us prove that from (2) and (3) it follows that;

$$a_n < t\left(t\left(t\left(t\left(F_{u,Hv}\left(\frac{r_5}{a}\right), F_{v,Hu}\left(\frac{r_4}{b}\right)\right), F_{v,Hv}\left(\frac{r_3}{c}\right)\right), F_{u,Hu}\left(\frac{r_2}{d}\right)\right), F_{u,v}\left(\frac{r_1}{e}\right).$$

First, we shall show that for everv $x \in (a_n, 1): T_s(x) > a_n$ for every $s \in N$ by induction. Suppose that $s = 1$. If $t = \min$ or there exists $b \in (0, 1)$ such that $t \mid [b, 1] = \min$, using the definition of the sequence $\{a_n\}_{n\in N}$ we conclude that $t(x, x) = x$ and so $T_1(x) = t(x, x) = x > a_n$. If there is no $b \in (0, 1)$ such that $t \mid [b, 1] = \min$ then $t(a_n, a_n) = a_n = c_{m(n)}$ and choose $\bar{\alpha}, \bar{\bar{\alpha}}$ from the interval $J_{m(n)} = (c_{m(n)}, b_{m(n)})$ such that $a_n < \bar{\alpha} < \bar{\bar{\alpha}} < x$. Since $t_{m(n)}$ is strict semigroup it folows that:

$$a_n = t(a_n, a_n) \leqslant t(\bar{\alpha}, \bar{\alpha})) < t(\bar{\alpha}, \bar{\bar{\alpha}})) \leqslant t(x, x)$$

and so $T_1(x) > a_n$ is all cases. Suppose now that $T_s(x) > a_n$ for some $s \in N$ and prove that $T_{s+1}(x) > a_n$. Since we proved that $y > a_n$ implies that $t(y, y) > a_n$ it follows that;

$$a_n < t(T_s(x), T_s(x)) \tag{4}$$

Further T-norm min is the strongest T-norm and so for every $m \in N$ we have that $T_m(x) \leqslant \min \{\underbrace{x, x, \ldots, x}_{n\text{-times}}\} = x$. Using (4) we conlucude that;

$$a_n < t(T_s(x),\ T_s(x)) < t(T_s(x),\ x) = T_{s+1}(x)$$

Because from (2) and (3) it follows that;

$$a_n < \min\left\{F_{u,v}\left(\frac{r_1}{e}\right),\ F_{u,Hu}\left(\frac{r_2}{d}\right),\ F_{v,Hv}\left(\frac{r_3}{c}\right),\ F_{u,Hu}\left(\frac{r_4}{b}\right),\ F_{u,Hv}\left(\frac{r_5}{a}\right)\right\}$$

and so;

$$a_n < T_4\left(\min\left\{F_{u,v}\left(\frac{r_1}{e}\right),\ F_{u,Hu}\left(\frac{r_2}{d}\right),\ F_{v,Hv}\left(\frac{r_3}{c}\right),\ F_{v,Hu}\left(\frac{r_4}{b}\right),\ F_{u,Hv}\left(\frac{r_5}{a}\right)\right\}\right),$$

From this it folows that;

$$a_n < t\left(t\left(t\left(t\left(F_{u,Hv}\left(\frac{r_5}{a}\right),\ F_{v,Hu}\left(\frac{r_4}{b}\right)\right),\ F_{v,Hv}\left(\frac{r_3}{c}\right)\right),\ F_{u,Hu}\left(\frac{r_2}{d}\right)\right),\right.$$

$$\left. F_{u,v}\left(\frac{r_1}{e}\right)\right) \leqslant F_{Hu,Hv}\left(\sum_{i=1}^{5} r_i\right) \leqslant a_n$$

which is a contradiction.

Now we have;

$$d_n(Hu, Hv) \leqslant e d_n(u, v) + d \cdot d_n(u, Hu) + c \cdot d_n(v, Hv) + b \cdot d_n(v, Hu)$$
$$+ a \cdot d_n(u, Hv) \leqslant (e + d + c + b + a) \max\{d_n(u, v)\, d_n(u, Hu),\ d_n(v, Hv),$$
$$d_n(v, Hu),\ d_n)u, Hv)\}$$

and using the same method as Lj. Ćirić in [5] it follows that there exists one and only one fixed point x^* of the mapping H and $\lim_{n\to\infty} H^n x_0 = x^*$ which completes the proof.

Corollary [15] *Let* $(S, \mathcal{F}, \min)$ *be a complete Menger space and* $H: S \to S$ *be such that for every* $r > 0$ *and every* $u, v \in S$:

$$F_{Hu,Hv}(q \cdot r) \geqslant F_{u,v}(r) \text{ where } q \in (0, 1)$$

Then there exists one and only one fixed point x^* *of the mapping* H *and* $\lim_{n\to\infty} H^n x_0 = x^*$ *for every* $x_0 \in S$.

Proof; If $a, b, c, d > 0$ are such that $a + b + c + d + q < 1$ then from:

$$F_{Hu,Hv}(r_1 + r_2 + r_3 + r_4 + r) \geqslant F_{Hu,Hv}(r) \geqslant F_{u,v}\left(\frac{r}{q}\right) =$$

$$= t(t(t(t(1, 1), 1), 1),\ F_{u,v}\left(\frac{r}{q}\right) \geqslant t\left(t\left(t\left(t\left(F_{u,Hv}\left(\frac{r_4}{a}\right),\ F_{v,Hu}\left(\frac{r_3}{b}\right)\right),\right.\right.\right.$$

$$\left.\left.\left. F_{v,Hv}\left(\frac{r_2}{c}\right)\right),\ F_{u,Hu}\left(\frac{r_1}{d}\right)\right),\ F_{u,v}\left(\frac{r}{q}\right)\right)$$

we conlucude that all the conditions of Theorem are satisfied and so there exists one and only one fixed point x^* of the mapping H and $\lim_{n\to\infty} H^n x_0 = x^*$, where $x_0 \in S$ is an arbitrary element of S. Now, we shall give an example of T-norm t such that the family $\{T_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$ and that t_k is strict for every $k \in K$.

Let $\bar{t}(x, y) = x \cdot y$ for every $x, y \in [0, 1]$ and let us define T-norm t in the following way;

$$t(x, y) = \begin{cases} 1 - 2^{-m} + 2^{-m-1}\,\bar{t}\,(2^{m+1}(x-1+2^{-m}),\ 2^{m+1}(y-1+2^m)) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } (x, y) \in J_m \times {}_xJ^m \\ \min\{x, y\} \qquad\qquad\qquad\qquad\qquad\qquad \text{if } (x, y) \notin \bigcup_{m=0}^{\infty} J_m \times {}_xJ^m \end{cases}$$

where $J_m = [1 - 2^{-m},\ 1 - 2^{-m-1}[$ for $m = 0, 1, 2, \ldots$. It is easy to see that the family $\{T_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$ since $\lim_{n\to\infty} 1 - 2^{-m} = 1$.

In [4] Lj. Ćirić proved the following fixed point theorem; *Let* $(S, \mathcal{F}, \min)$ *be a complete Menger space,* $T: S \to S$ *and* $q \in (0, 1)$ *such that for every* $u, v \in S$;

$$F_{Tu,\,Tv}(qx) \geqslant \min\{F_{u,\,v}(x),\ F_{u,\,Tu}(x)\,F_{v,\,Tv}(x),\ F_{u,\,Tv}(2x),\ F_{v,\,Tu}(2x)\} \tag{5}$$

fos every $x > 0$. *Then there exists one and only one fixed point* x^* *of the mapping* T *and* $x^* = \lim T^n x_0$ *for every* $x_0 \in S$.

Using the method from [3] we shall give a short proof of this theorem.

Proof: Let $a \in (0, 1)$ and $d_a : S \times S \to R^+ \cup \{0\}$ be defined in the following way; $d_a(x, y) \mid F_{x,\,y}(t) \leqslant 1 - a\}$. In [3] is proved that d_a is pseudometric and that the family $\{d_a\}_{a\in(0,\,1)}$ defines the (ε, λ)-topology. We shall show that for every $a \in (0, 1)$ we have that;

$$d_a(Tu, Tv) \leqslant q \max\left\{d_a(u, v),\ d_a(u, Tu),\ d_a(v, Tv),\ \frac{1}{2} d_a(u, Tv),\ \frac{1}{2} d_a(v, Tu)\right\}$$

for every $u, v \in S$. Suppose on the contrary that there exist $a \in (0, 1)$ and $u, v \in S$ such that;

$$d_a(Tu, Tv) > q \max\left\{d_a(u, v),\ d_a(u, Tu),\ d_a(v, Tv),\ \frac{1}{2} d_a(u, Tv),\ \frac{1}{2} d_a(v, Tu)\right\}$$

Then there exists $r > 0$ such that;

$$d_a(Tu, Tv) > r > q \max\left\{d_a(u, v),\ d_a(u, Tu),\ d_a(v, Tv),\ \frac{1}{2} d_a(u, Tv),\ \frac{1}{2} d_a(v, Tu)\right\}.$$

Using the definition d_a from this it follows that;

$$F_{Tu,\,Tv}(r) \leqslant 1 - a \quad \text{and} \quad \min\left\{F_{u,\,v}\left(\frac{r}{q}\right),\ F_{u,\,Tu}\left(\frac{r}{q}\right),\ F_{v,\,Tv}\left(\frac{r}{q}\right),\right.$$
$$\left. F_{u,\,Tv}\left(\frac{2r}{q}\right) F_{v,\,Tu}\left(\frac{2r}{q}\right)\right\} > 1 - a$$

which is a contradiction. Similarly as in [5] it follows the above cited Ćirić's fixed point theorem.

Remark; In Ćirić's fixed point theorem [4] we can replace the inequality (5) by the following weaker condition; For every $u, v \in S$ it follows;

$$F_{Tu, Tv}(qx) \geqslant \min\{F_{u, v}(x),\ F_{u, Tu}(x)\, F_{v, Tv}(x),\ F_{u, Tv}(x),\ F_{v, Tu}(x)\} \tag{6}$$

The proof is similar as the above proof of Ćirić's result from [4] since from (6) it follows; $d_a(Tu, Tv) \leqslant 2 \max\{d_a(u, v)\, d_a(u, Tu)\, d_a(u, Tv),\ d_a(v, Tv),\ d_a(v, Tu)\}$ and then we can apply Ćirić's result from [5] (after slight modification).

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Olga Hadžić
Faculty of Sciences
Department of Mathematics
21000 NOVI SAD
Dr Ilije Đuričića 4
Yugoslavia