PUBLICATIONS DE L'INSTITUT MATHEMATIQUE
Nouvelle série, tome 26 (40), 1979, pp. 91–96

FIXED POINTS OF ANTIMORPHISMS

R. Dacić

(Received June 9, 1979)

Introduction

O. Let P be a non-void set and $f:P\to P$. The mapping f has a fixed point if there exists an element a in P such that $f(a)=a$. If P is a partially ordered set and $f:P\to P$ has the poperty

(1) $$a, b\in P \text{ and } a\leqslant b \text{ imply } f(a)\leqslant (b),$$

then f is said to be *isotone* (or increasing).

If f satisfies the condition

(2) $$a, b\in P \text{ and } a\leqslant b \text{ imply } f(a)\geqslant f(b),$$

then f is said to be *antitone* (or decreasing). Since Tarski's famous theorem [6], that every isotone mapping of a complete lattice into itself has a fixed point, many papers appeared concerning fixed points of mappings in partially ordered sets. Among them some papers deal antitone mappings (see [1] to [5]). In this note we treat two kinds of mappings (defined by (3) and (4) bellow) of a complete lattice into itself and obtain results which generelize results of [5].

Let L be a complete lattice. The mapping $f:L\to L$ satisfying the condition

(3) $$f(\inf M)=\sup f(M) \text{ for any } \varnothing \neq M\subset L$$

where $f(M)=\{f(x)\mid x\in M\}$ and $\inf M$ (resp. $\sup M$) is another notation for $\wedge\{x\mid x\in M\}$ (resp. $\vee\{x\mid x\in M\}$) is called a *meet antimorphism.*

Similarly, if f satisfies the condition

(4) $$f(\sup M)=\inf f(M), \quad \text{for } \varnothing \neq M\subset L,$$

then such an f is said to be a *join antimorphism.*

It is easily seen that if f, defined in a complete lattice L and satisfying (3) and (4), (we call such mappings *antimorphisms*), then it must also satisfy (2).

Let us note that we shall write $a \wedge b$ (resp. $a \vee b$) and not $\inf\{a, b\}$ (resp. $\sup\{a, b\}$).

In this paper we examine fixed points of such antitone mappings $f: L \to L$ which are comparable to the identity mapping $i: L \to L$, in the sense that for any $x \in L$ is $f(x) \leqslant x$ or $f(x) \geqslant x$. It appears that such mappings are antimophisms. More precisely, the following proposition is valid.

Proposition O. *Let L be a complete lattice and $f: L \to L$ an antitone mapping satisfying the condition: $(\forall x \in L)(f^2(x) \leqslant x)$. Then f is a meet antimorphism.*

Remark. $f^2 = f \circ f$.

Proof. Let $\varnothing \neq A \subset L$ and $m = \inf A$. Then $f(m) \geqslant f(x)$ for every $x \in A$, i.e. $f(m)$ is an upper bound for $f(A)$. Let $s = \sup f(A)$. Evidently

$$s \leqslant f(m). \tag{α}$$

From $f(x) \leqslant s$, it follows $f^2(x) \geqslant f(s)$, or (since $x \geqslant f^2(x)$) $x \geqslant f(s)$. We conclude that $f(s)$ a lower bound for A. Then $f(s) \leqslant m$, which implies $f^2(s) \geqslant f(m)$, or

$$s \geqslant f(m) \tag{β}$$

Our conclusion follows from (α) and (β), that is $f(\inf A) = \sup f(A)$.

Analogous statement for join antimorphism is also valid.

1. Fixed points of meet antimorphisms

Theorem 1. *Let L be a complete lattice satisfying the following condition:*

(α) *There exists an antichain $A \subset L$ such that every $x \in L$ is either a join or a meet of elements of A.*

Let $f: L \to L$ be an antitone mapping such that

(i) *For every $x \in L$ is $f^2(x) \leqslant x$;*

(ii) *for every $x \in A$ is $f(x)$ comparable to x;*

(iii) $\{x \in A \mid x \geqslant f(x)\} \neq \varnothing$.

Then there exists a fixed point of f.

Proof. Denote be $P_f(L)$ the family of all subsets B of L satisfying $f(\inf B) \leqslant \inf B$. By (iii) $P_f(L)$ is nonempty.

Lemma. *Under the assumption of the theorem $P_f(L)$, ordered by inclusion, has a maximal element.*

Proof of the lemma. Let $\{B_i \mid i \in I\}$ be a chain in $(P_f(L), \subseteq)$ and put $\inf B_i = b_i$, $B_0 = \cup\{B_i \mid i \in I\}$. We shall show that $B_0 \in P_f(L)$. Let $i, j \in I$. We may assume $B_i \leqslant B_j$ and so $b_i \geqslant b_j \geqslant f(b_j) \geqslant f(b_i)$. It follows that $\inf\{b_j \mid j \in I\} \geqslant f(b_i)$ for each $i \in I$, and so $\inf\{b_j \mid j \in I\} \geqslant \sup\{f(b_i) \mid i \in I\}$. Since f is, by proposition O, a meet antimorphism, we obtain

$$\inf\{b \mid b \in B_0\} = \inf\{b_i \mid i \in I\} \geqslant \sup\{f(b_j) \mid j \in I\} =$$

$$f(\inf\{b_i \mid j \in I\}) = f(\inf B_0), \text{ proving that } B_0 \in P_f(L).$$

The assertion of the lemma follows now from Zorn's lemma.

Proof of the theorem. Put

$$A_f=\{a\in A \mid a\geqslant f(a)\}$$
$$A^f=\{a\in A \mid a<f(a)\}.$$

By the lemma, there exists $B_0\in P_f(L)$ such that $f(b_0)\leqslant b_0$, where $b_0=\inf B_0$.

If $f(b_0)=b_0$ we are done. Suppose $f(b_0)<b_0$. Then, since L satisfies the condition (α), there exists $a\in A$, a non-comparable to b_0, such that

(1) $$f(b_0)\leqslant b_0\wedge a.$$

We distinguish two cases.

Case 1° $a\in A_f$. Then $f(b_0)\leqslant a$ implies $f(a)\leqslant f^2(b_0)\leqslant b_0$. From this and $f(a)\leqslant a$ (since $a\in A_f$) we obtain

(2) $$f(a)\leqslant b_0\wedge a.$$

The relations (1) and (2) imply

$$f(b_0)\vee f(a)\leqslant b_0\wedge a.$$

But $f(b_0)\vee f(a)=f(b_0\wedge a)$, hence $f(b_0\wedge a)\leqslant b_0\wedge a$, contradicting maximality of B_0.

Case 2°. $a\in A^f$. Then (1) yields

$$f^2(b_0)\geqslant f(b_0)\vee f(a)$$

or, by (i)

$$b_0\geqslant f(b_0)\vee f(a)\geqslant f(a)>a,$$

contradicting the hypothesis that b_0 and a are incomparable.

The hypothesis $f(b_0)<b_0$ must be rejected, and (since $f(b_0)\leqslant b_0$)$f(b_0)=b_0$ follows.

Remark 1. The following will show that neither of the conditions of theorem 1 may be individually dropped.

Example 1. Let L be the lattice on the figure 1 and $f\colon L\to L$ defined by

$$f=\begin{pmatrix}0 & a & b & 1\\ 1 & 1 & 0 & 0\end{pmatrix}$$

The conditions (ii) and (iii) are fulfilled, but not (i). No fixed point of f exists.

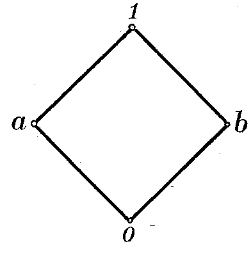

Figure 1

Example 2. The same lattice as in example 1 with

$$f=\begin{pmatrix}0 & a & b & 1\\ a & 0 & a & 0\end{pmatrix}.$$

The conditions (i) and (iii) are fulfilled, but not (ii). There is no fixed point of f.

Example 3. The same lattice as in example 1 with f defined by

$$f=\begin{pmatrix}0 & a & b & 1\\ 1 & 1 & 1 & 0\end{pmatrix}.$$

The conditions (i) and (ii) are fulfilled but not (iii). No fixed points of f exists.

Remark 2. If the complete lattice L is coatomic, i.e. if every $x\in L$, $x\neq 1$, is a meet of coatoms — elements of L covered by 1, then, if by C is denoted the set of coatoms, $C_f=\varnothing$ implies $f(x)=1$ for every $x\in L$, so that the condition (iii) can be omitted. In this case we have the following

Corollary 2. *Let L be a complete coatomic lattice and $f:L\to L$ an antitone mapping such that*

(i) $f^2(x)\leqslant x$ *for every* $x\in L$:

(ii) $f(a)$ *is comparable to* a *for every* $a\in C$.

Then f has a fihed point.

2. Fixed points of join antimorphisms

Let L be a complete lattice and $f:L\to L$ an antitone mapping. Then the following theorem is valid.

Theorem 3. *Let L be a complete lattice satisfying the condition* (α) *of theorem 1 and let $f:L\to L$ be an antitone mapping such that*:

(i) *for every* $x\in L$ *is* $f^2(x)\geqslant x$;

(ii) *for every* $a\in A$ *is* $f(a)$ *comparable to* a;

(iii) $\{a\in A\mid a\leqslant f(a)\}\neq\varnothing$.

Then f has a fixed point.

The proof of this theorem is similar to the proof of theorem 1 and will be omitted.

Examples similar to the examples following theorem 1 can show that neither of the conditions of theorem 3 may be individually dropped.

Remark 3. If the complete lattice L is atomic, i.e. if every $x\in L$, $x\neq 0$, is a join of atoms-elements of L that cover O, then (if by A is denoted the set of atoms) $A^f=\varnothing$ implies $f(x)=0$ for every $x\in L$, so that the condition (iii) can be dropped. In this case we have the following

Corollary 4. *Let L be a complete atomic lattice and let $f:L\to L$ be a an antitone mapping such that*:

(i) *for every* $x\in L$ *is* $f^2(x)\geqslant x$;

(ii) $f(a)$ *is compparable to* a *for every* $a\in A$;

Then f has a fixed point.

Corollary 5. ([5], *Theorem* 1) *Let L be a complete atomic lattice and let $f:L\to L$ be an antitone mapping satisfying the cndition* (i) *of corollary* 4. *If $f(a)\geqslant a$ for each atom a, then f has a fixed point.*

3. Commuting mappings

Theorem 6. *Let L be a complete lattice satisfying the condition* (α) *of theorem* 1 *and let $f, g:L\to L$ be two antitone mappings such that:*

(i) *for every $x\in L$ is $f^2(x)\leqslant x$, $g^2(x)\leqslant x$;*

(ii) *for every $a\in A$ is $f(a)$ comparable to a, $g(a)$ is comparable to a;*

(iii) $A_f=A_g\neq\varnothing$;

(iv) $fg=gf$.

Then f and g have a common fixed point.

Proof. If we put $(f\vee g)(x)=f(x)\vee g(x)$, then $(f\vee g)^2(x)=(f\vee g)(f(x)\vee g(x))=f(f(x)\vee g(x))\vee g(f(x)\vee g(x))\leqslant f^2(x)\vee g^2(x)\leqslant x$, so that $f\vee g$ fulfills the condition (iii) of theorem 1. Evidently $f\vee g$ is a meet antimorphism.

Since $f\vee g$ fulfills all the conditions of theorem 1, there exists a point b_0 such that $(f\vee g)(b_0)=b_0$. We know that $b_0\leqslant a$ for some $a\in A$.

From $f(b)\leqslant b_0$, $g(b_0)\leqslant b_0$ we obtain $(fg)(b_0)\geqslant g(b_0)$, $(gf)(b_0)\geqslant f(b_0)$, so that

$$(fg)(b_0)\geqslant f(b_0)\vee g(b_0)=b_0. \tag{1}$$

The condition (i) implies

$$f^2(g(b_0))=g(b_0),\ g^2(f(b_0))=f(b_0). \tag{2}$$

From (1) it follows $f^2(g(b_0))\leqslant {}^f(b_0)$ and $g^2(f(b_0))\leqslant g(b_0)$. This and (2) imply $f(b_0)\leqslant g(b_0)\leqslant f(b_0)$, or $f(b_0)=g(b_0)=(f\vee g)(b_0)=b_0$, proving the theorem.

Corollary 7. *Let L be a complete coatomic lattice with the set C of coatoms an let $f, g:L\to L$ be two antitone mappings satisying conditions* (i) — (iv) *of the theorem* 6. *Then f and g have a common fixed point.*

Theorem 8. *Let L be a complete lattice satisfying the condition* (α) *of theorem* 1 *and let $f, g:L\to L$ be two antione mappings such that:*

(i) *fos every $x\in L$ is $f^2(x)\geqslant x$, $g^2(x)\geqslant x$;*

(ii) *for every $a\in A$ is $f(a)$ comparable to a, and $g(a)$ is comparable to a;*

(iii) $A^f=A^g\neq\varnothing$;

(iv) $fg=gf$.

Then f and g have a common fixed point.

Proof of this theorem is analogous to the proof of theorem 6 and will be omitted.

Corollary 9. *Let L be a complete atomic lattice (with the set A of atoms) ant let $f, g:L\to L$ be two antitone mappings satisfying the conditions* (i) — (iv) *of the theorem* 8. *Then f and g have a common fixed point.*

C o r o l l a r y 10. ([5], *theorem* 2) *Let L be a comlete atomic lattice and let* $f, g: L \to L$ *be antitone mappings which satisfy the condition* (i) *of the theorem* 8 *and* $f(a) \geqslant a$, $g(a) \geqslant a$ *for each atom a. If* $fg = gf$, *then f and g possess a common fixed point.*

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