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FIXED POINTS OF ANTITONE MAPPINGS IN CONDITIONALLY COMPLETE PARTIALLY ORDERED SETS

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In the first part of this paper we study some general questions concerning fixed points of antitone self-mappings of a partially ordered set. In the second part is assumed that the underlying set is conditionally complete (dense or not) and the mappings are still antitone.

0. Introductory concepts and notation

Let P be any nonvoid set and $f: P \to P$. We donote by $I(f, P)$ the set of all fixed points of f, i.e.

$I(f, P) = \{x \mid x \in P \text{ and } f(x) = x\}$.

If $\leqslant$ is an order relation on P, i.e. if $(P, \leqslant)$ is an ordered set, we use the following notation, borrowed from Kurepa (see [3]):

$$P^f = \{x \mid x \in P \text{ and } x \leqslant f(x)\}$$

$$P_f = \{x \mid x \in P \text{ and } x \geqslant f(x)\}$$

Evidently, $I(f, P) = P^f \cap P_f$.

Instead of $f \circ f$ we shall write f^2. More generally, we use f^n for $f \circ f^{n-1}$, where n is a positive integer and f^0 the identity mapping.

Let P be a partially ordered set (poset) and $f: P \to P$.

A function f will be called *isotone* if a, $b \in P$ and $a \leqslant b$ imply

$$f(a) \leqslant f(b) \tag{1}$$

and *antitone* if $a \leqslant b$ implies

$$f(a) \geqslant f(b). \tag{2}$$

In complete lattices L are considered functions f such that, for any $\varnothing \neq A \subset L$

$$f(\vee A) = \wedge f(A), \quad \text{where} \quad f(A) = \{f(a) \mid a \in A\}. \tag{3}$$

A function f in a complete lattice L satisfying (3) is referred to as *join antimorphism.*

One considers also *meet antimorphism*, satisfying

$$f(\wedge A)=\vee f(A), \quad \varnothing\neq A\subset L. \tag{4}$$

It is easily seen that every funotion f, defined on a complete lattice L, satisfying (3) and (4) also satisfies (2), that is, join and meet antimorphisms are antitone mappings.

On the other hand, it is easy to construct an antitone mapping on a complete lattice which is neither a join antimorphism nor a meet antimorphism (see Example 1).

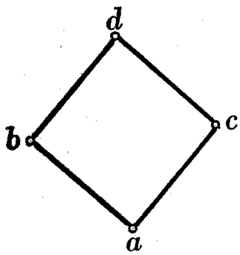


Figure 1

Example 0.1. Let P be the lattice on the figure 1 and $f: P\to P$ defined by:

$$f(\{a, b, c\})=\{d\}, \quad f(d)=a.$$

Evidently, f is antitone, but

$$f(\sup\{a, b, c\})=f(d)=a\neq$$

$$\inf\{f(a), f(b), f(c)\}=\inf\{d\}=d.$$

Mapping f is not a join antimorphism.

If we define $g: P\to P$, $g(\{b, c, d\})=\{a\}$ and $g(a)=d$, then g is antitone, but not a meet antimorphism.

One can construct an antitone mapping of a complete lattice L into itself, which is neither a join antimorphism nor a meet antimorphism.

1. Some general properties of antitone mappings

The famous Tarski's theorem (see [5]) asserts that any isotone mapping of a complete lattice L into itself has a non-void set I of fixed points and this set is a complete sublattice of L relative to the same partial order.

Unlike of isotone, an antitone mapping of a complete lattice into itself can be without any fixed point. But if the set of fixed points of an antitone mapping is non-void and contains more than one point, then this set is never a sublattice of L. Moreover it is true the following

Proposition 1.1. *Let $(P, \leqslant)$ be a partially ordered set, $f: P\to P$ antitone and $I(f, P)\neq\varnothing$. Then $(I(f, P), \leqslant)$ is an antichain of $(P, \leqslant)$.*

If $I(f, P)$ is a singleton then there is nothing to prove. Suppose there are at least two different fixed point of f, x and y say, and $x<y$. Applying f we obtain $f(x)\geqslant f(y)$, or, since both points are fixed under f, $x\geqslant y$, which contradicts the supposition $x<y$.

In connection with the proposition 1.1 natural questions arise:

Question 1.1. Given a partially ordered set P and an antichain $A\subset P$. Under which conditions there exists an antitone mapping $f: P\to P$ such that the only fixed point of f are just the elements of A?

In the case when P is a chain the answer is affirmative.

Question 1.2. In cases in which the answer to the question 1.1 is affirmative, does there exist an antitone mapping which is $1:1$?

For a chain, the answer to the question 1.2 is affirmative only if the point a, where $A=\{a\}$ is the given antichain, is some „center" of the chain.

Factorisation. Definition. Let $f, f_1, f_2: P \to P$. The mapping f is said to be *factorable* if $f=f_1 \circ f_2$. If P is a poset and f_1, f_2 antitone self-mappings of P, then, as can be easily seen, $f_1 \circ f_2$ is an antitione self-mapping of P.

Let f be an arbitrary isotone self-mapping of a poset P. If there exist two antitone self-mappings of P, f_1 and f_2 say, such that $f=f_1 \circ f_2$, then we say that f is *2-factorable.*

In the case $f_1=f_2$, then f is said to be 1-factorable.

Not every isotone self-mapping of a poset P can be 1-factorable as shows the following example.

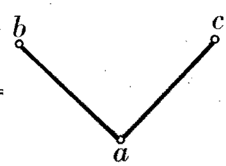
Example 1.1. If $P=$ 

Figure 2

and f is the identity mapping, then f cannot be 1-factorable.

Every constant mapping $f: P \to P$ is 1-factorable as $f=f \circ f$. Such a mapping is said to be self-factorable.

What about fixed points of f, f_1 and f_2 provided that $f=f_1 \circ f_2$?

If f_1 and f_2 have a common fixed point, then evidently the common fixed point of f_1 and f_2 is also a fixed point of f. But the existence of a fixed point of f does not imply the existence of a common fixed point of f_1 and f_2.

Theorem 1.1. *Let P be a poset, $f: P \to P$, $f=f_1 \circ f_2=f_2 \circ f_1$ and $|I(f, P)|=1$. Then f_1 and f_2 have the unique common fixed point.*

Proof. Let $a \in I(f, P)$. Then $f_1(a)=f_1 \circ (f_1 \circ f_2)(a)=f_1 \circ (f_2 \circ f_1)(a)= =(f_1 \circ f_2) \circ f_1(a)=(f \circ f_1)(a)=f(f_1(a))$, i.e. $f_1(a) \in I(f, P)$. But $I(f, P)$ is a singleton, hence $f_1(a)=a$.

Similarly $f_2(a)=a$.

Suppose now f_1 and f_2 have another common fixed point, b say. Then $b=f_1(b)$ implies $f_2(b)=f(b)=b$, contradicting the hypothesis $|I(f, P)|=1$.

Corollary 1.1. *Let P be any poset, $f: P \to P$ an antitone mapping and $|I(f^2, P)|=1$. Then f has a unique fixed point.*

The corollary 1 has the following generalisation, due to D. Adamović (see Matematički vesnik 8 (23), 1971, problem 236.).

Proposition 1.2. *Let P be any non-void set and $f: P \to P$ such a mapping that for some natural n f^n has a unique fixed point. Then the mapping f has a unique fixed point as well.*

Proof. Let $s=f^n(s)$. Then $f(s)=f(f^n(s))=f^n(f(s))$. Hence $f(s)\in I(f^n, P)$. This and $s\in I(f^n, P)$ and $|I(f^n, P)|=1$ imply $f(s)=s$. The unicity of the fixed point of f is obvious.

2. Antitone self-mappings of conditionally complete posets

In this section we assume that $(P, \leqslant)$ is a conditionally complete poset. A poset P is said to be *conditionally complete* if every non-empty subset of P which is bounded has a supremum and an infimum in P. A poset P is *dense* if, for any $x, y\in P$, $x<y$ implies the existence of $z\in P$ such that $x<z<y$.

Proposition 2.1. *If P is an arbitrary non-void poset and $f:P\to P$ an antitone mapping, then*

$$P^f\neq\varnothing \Leftrightarrow P_f\neq\varnothing.$$

Proposition 2.2. *With hypothesis on P and f like in proposition* 1, *the following two conditions are equivalent:*

(i) *Every point of P^f is comparable to every point of P_f;*

(ii) *$P^f\leqslant P_f$ (i.e. for every $x\in P^f$ and every $y\in P_f$ is $x\leqslant y$.)*

Proposition 2.3. *Let P be a conditionally complete poset, $f:P\to P$ antitone. If, furthermore, $P^f\leqslant P_f$* (both set being nonempty), *then* $\sup P^f$ (denote it by s) *and* $\inf P_f$ (denote it by i) *both exist, and $s\leqslant i$.*

If moreover P is dense, then $s=i$.

Remark 2.1. The converse of the proposition 3 is not true, i.e. if P is a conditionally complete, poset $f:P\to P$ antitone, $i=\inf P_f$, $s=\sup P^f$ exist, then the condition $P^f\leqslant P_f$ need not be valid. This shows the following.

Example 2.1.

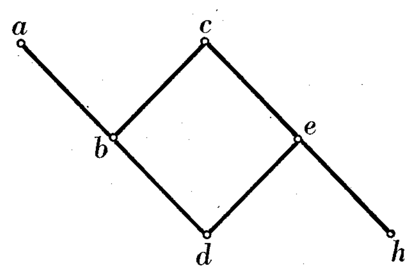
$P=$ 

Figure 3

$f:P\to P$ is defined by:

$f(a)=f(b)=f(c)=d,$

$f(d)=f(e)=f(h)=c.$

If is easily verified that f is antitone, $P_f=\{a, b, c\}$, $P^f=\{d, e, h\}$, $\sup P^f=e$, $\inf P_f=b$, but b is not comparable to e.

Proposition 2.4. *If P is non empty poset, $f:P\to P$ antitone and $P^f\leqslant P_f$, then*

$$f(P_f)\subset P^f,\quad f(P^f)\subset P_f.$$

Proposition $1-4$ are either parts of theorems or parts of proofs of theorems in paper [3], and their proof will be omitted.

Proposition 2.5. *Let P be a conditionally complete poset, $f: P \to P$ antitone and, for $\varnothing \neq A \subset P$, $\sup A$ ($\inf A$) exists. Then $\inf f(A)$ ($\sup f(A)$) also exists (where $f(A)$ is the* set $\{f(x) \mid x \in A\}$).

Proof. Let $a = \sup A$ ($b = \inf A$). Then for every $x \in A$ we have $x \leqslant a$ ($b \leqslant x$). Since f is antitone, it follows $f(x) \geqslant f(a)$ ($f(b) \geqslant f(x)$). Hence, the set $S = \{f(x) \mid x \in A\}$ is bounded from below (above) and $\inf f(A)$ ($\sup f(A)$) exists, by conditionally completeness of P.

The notions of join antimorphism and meet antimorphism can be defined in every conditionally complete poset, as follows.

A mapping $f: P \to P$ is a join (meet) antimorphism if for every $\varnothing \neq A \subset P$ bounded from above (from below), $f(A)$ is bounded from below (from above) and the relation (3) (resp. (4)) of the section O is valid.

Proposition 2.6. *Let $(P, \leqslant)$ be a conditionally complete nonempty poset, $f: P \to P$ a join antimorphism, $P^f \leqslant P_f$* (both sets being nonempty) *and $s \in P_f$. Then $s = i$.*

Remark 2.2. In the sequel we use the notation $s = \sup P^f$, $i = \inf P_f$.

Proof. One see easily that f is antitone. Hence propositions 1—5 are valid. According to proposition 3, $s \leqslant i$. But $s \in P_f$ and, since $i = \inf P_f$, $s \geqslant i$.

Proposition 2.7. *Let $(P, \leqslant)$ be a conditionally complete nonempty poset, $f: P \to P$ a meet antimorphism, $P^f \leqslant P_f$ (both sets being nonempty) and $i \in P^f$. Then $s = i$.*

Proof is analogous to the proof of the proposition 2.6.

Remark 2.3. The proposition 2.6 and 2.7. are valid for an arbitrary antitone mapping but we need them such, as they are stated.

Theorem. 2.1. *Let P be conditionally complete non-empty poset, $f: P \to P$ a join antimorphism and a meet antimorphism, $P^f \leqslant P_f$ and $s = i$. Then $f(s) = s$.*

Proof. Since f is a join antimorphism and $s = i$, we have $f(s) = f(\sup P^f) = \inf f(P^f) \geqslant$ (by proposition 4 and 3) $i = s$.

But f is also a meet antimorphism, hence

$f(s) = f(i) = f(\inf P_f) = \sup f(P^f) =$ (by proposition 4 and 3) $\leqslant s$.

The following simple example shows that the condition $s = i$ in the theorem 2.1 cannot be removed.

Example 2.2.

$P =$ 

$f(a) = b, \quad f(b) = a.$

Figure 4

The mapping f so defined is both a join antimorphism and a meet antimorphism, $P^f \leqslant P_f$, but $s < i$ and no fixed point of f exists. In Abian's theorem [1] this situation is removed by hypothesis that P is dense.

Theorem 2.2. *Under the hypothesis of the proposition* 2.6, $f(s) = s$.

Proof. By proposition 2.6. $s = i$. The rest of the proof runs like the first part of the proof of theorem 2.1.

Theorem 2.3. *Under the hypothesis of proposition* 2.7. *is* $f(s) = s$.

Proof applies the proposition 2.7. and runs like the second part of the proof of the theorem 2.1.

Remark 2.4. The theorem 2.2. is essentially reformulated Kurepa's theorem 2 (see [3]). The following example shows that the condition $s \in P_f$ cannot be removed, even in the case of a finite lattice.

Example 2.3. Let

$$f(\{d, g, e, h\}) = \{a\}, \quad f(x) = b,$$

$$f(b) = c, \quad f(a) = d.$$

$P =$

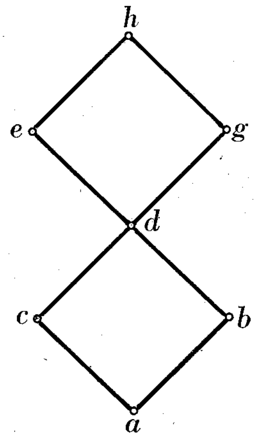


Figure 5

All conditions of the theorem 2.2. are fulfilled excepting $s \in P_f$, but no fixed point of f exists.

The following two theorems make more precise the theorem 2:16 of [3].

Theorem 2.4. *Let* $(P, \leqslant)$ *be a conditionally complete nonempty poset,* $f: P \to P$ *a join antimorphism such that there exist nonempty sets* X *and* Y *such that* $X \subset P^f$, $Y \subset P_f$, $f(X) \subset Y$, $f(Y) \subset X$,

$$\sup X = \inf Y = z$$

and $z \in P_f$. *Then* $f(z) = z$.

Theorem 2.5. *All is same as in the theorem* 2.4. *with a meet antimorphism instead of a join antimorphism and* $z \in P^f$ *instead of* $z \in P_f$.

Proofs of these two theorems run like the proof of the theorem 2.1. and will be omitted.

R e m a r k 2.5. Evidently, the theorem 2.2. (resp. 2.3.) is a special case of the theorem 2.4. (resp. 2.5.). For $X=P^f$, $Y=P_f$ we obtain theorems 2.2. and 2.3.

T h e o r e m 2.6. *Let* $(P, \leqslant)$ *be a non-empty conditionally complete poset,* $f: P \to P$ *a mapping which is simultanously a join antimorphism and a meet antimorphism, and there exist non-empty sets* X *and* Y *such that* $X \subset P^f$, $Y \subset P_f$, $f(X) \subset Y$, $f(Y) \subset X$ *and* $\sup X = \inf Y = z$. *Then* $f(z) = z$.

Proof of this theorem is similar to the proof of the theorem 2.1. and will be omitted.

T h e o r e m 2.7. *Let* $(P, \leqslant)$ *be a conditionally complete nonvoid poset,* $f: P \to P$ *an antitone mapping such that*

(i) $P^f \leqslant P_f$,

(ii) $s \in P_f$, $i \in P^f$.

Then 1° $s = i$;

2° $f(s) = s$.

P r o o f. The condition (ii) means $f(s) = s$ and $f(i) = i$. This, the condition (i) and the proposition 2.2. imply

$$f(s) = s = i = f(i). \tag{*}$$

Since f is antitone and $s = i$, we have $f(i) \leqslant f(s)$. This and (*) imply $f(s) = f(i) = s = i$.

In papers [1] and [4] the underlying poset is assumed to be:

(i) totally ordered,

(ii) dense (for any $a, b \in P$, $a < b$ implies that there exists $c \in P$ such that $a < c < b$),

(iii) conditionally complete.

A poset P satisfying the conditions (i) to (iii) is said to be continuous.

In [4] the following theorem is proved.

T h e o r e m MP. *Suppose* $f: P \to P$ *satisfies*

(1) $[f(y), f(x)] \subset f([x, y])$ *whenever* $x, y \in P$, $x \leqslant y$ *and* $f(y) \leqslant f(x)$,

(2) f *is nonoscillatory either from the right or from the left,*

(3) $\exists\, a, b \in P$ *such that* $a \leqslant b$, $a \leqslant f(a)$, $f(b) \leqslant b$.

Then f *has a fixed point in the interval* $[a, b]$.

A function $f: P \to P$ is said to be nonoscillatory from the right if, for each $x \in P$,

$$\bigcap_{u > x,\ u \in P} f([x, u]) \text{ has at most one point.}$$

Likewise, f is said to be nonoscillatory from the left if, for each $x \in P$,

$$\bigcap_{u < x,\ u \in P} f([u, x]) \text{ has at most one point.}$$

We shall show that the condition (1) is sufficient to ensure the existence of a fixed point, if the poset satisfies the continuity condition and f is an antitone mapping. In other words we shall prove the following

Theorem 2.8. *Let P be a continuous poset and $f: P \to P$ an antitone mapping satisfying the condition*

(1) $x, y \in P$ *and* $x \leqslant y \Rightarrow [f(y), f(x)] \subset f([x, y])$.

Then f has the unique fixed point.

Proof. By proposition 1—4 we know that $s = \sup P^f$ and $i = \inf P_f$ both exist, $s = i$, and $f(P^f) \subset P_f$, $f(P_f) \subset P^f$.

Since every $x \in P$ is either in P_f or in P^f, we have $f(i) > i$ or $f(i) < i$, or $f(i) = i$.

Suppose $f(i) > i$.

Let $x \in (i, f(i)]$. Then $x \in P_f$, hence $f(x) \in P^f$, i.e. $f(x) \leqslant i$. In particular $f^2(i) \leqslant i$.

Hence, $f((i, f(i)]) \subset [f^2(i), i]$ and $f([i, f(i)]) \subset \{f(i)\} \cup [f^2(i), i]$.

On the other hand, by the condition (1)

$$[f^2(i), f(i)] \subseteq f([i, f(i)]) \subseteq [f^2(i), i] \cup \{f(i)\},$$

or

$$[f^2(i), i] \cup [i, f(i)] \subseteq [f^2(i), i] \cup \{f(i)\},$$

which implies

$$[i, f(i)] = \{i\}, \text{ i.e. } f(i) = i.$$

Similarly the hypothesis $f(i) < i$ is disproved.

Now we give a short proof of a recent result of A. Abian.

Theorem A. (*Abian*, [4]). *Let P be a nonempty simply ordered poset, which is conditionally complete, and $f: P \to P$ an antitone mapping.*

If, for every $x \in P$, $x \neq f(x)$ implies

(1°) $f(H) \cap H \neq \varnothing$, *where* $H = (x, f(x)]$ *or* $H = [f(x), x)$, *then f has a fixed point.*

Proof. Put $s = \sup P^f$. Suppose $s < f(s)$. Then $(s, f(s)] \subset P_f$, hence, according to the proposition 4, $f((s, f(s)]) \subset P^f \setminus P_f$, which contradicts (1°).

Similarly is disproved the supposition $s > f(s)$. Since P is simply ordered, it must be $s = f(s)$.

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