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ON ONE ESTIMATION PROBLEM

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1. Let $X=\{X(t),\ O\leqslant t\leqslant 1\}$ be a real-valued, purely nondeterministic random process of second order, continuous in quadratic mean. We denote by $H(X;\ t)$ $(H;\ t-0))$ the smallest Hilbert space spanned by random variables $X(s)$, $O\leqslant s\leqslant t$ $(O\leqslant s\leqslant t)$, and we put $H(X)=H(X;\ 1)$. From the mean square continuity of X it follows that the equality $H(X;\ t-0)=H(X;\ t)$ holds for any t. If $E_X(t)$ is a projection operator from $H(X)$ onto $H(X;\ t)$, then the family $E_X=\{E_X(t),\ O\leqslant t\leqslant 1\}$ forms a resolution of the identity in $H(X)$ [1]. The process X is supposed to have the unit multiplicity; the set of all processes on [0; 1] with the above properties we denote by C. It is well known [1] that every process X from C has a so-called Hida-Cramér representation

$$X(t)=\int_0^t g(t,\ u)\,dz(u),^{1),\,2)}\quad O\leqslant t\leqslant 1, \tag{1}$$

where the family $\{g(t,\ u),\ t\text{-parameter},\ u\leqslant t\}$ of non-random functions is complete in $L^2_{[0;\ 1]}(F_X)$ [8, 9], and F_X is the spectral type of X [1, 9], i.e.

$$F_X(t)=\|z(t)\|^2,\ O\leqslant t\leqslant 1.$$

In the sequel we shall be concerned with the following problem: if $X\in C$ iz known, what could be said about linear mean square estimation of the process $Y\in C$, when the condition

$$Y(t)\in H(X;\ t),\ O\leqslant t\leqslant 1 \tag{2}$$

is satisfied? It is clear that (2) implies

$$H(Y;\ t)\subset H(X;\ t),\ O\leqslant t\leqslant 1, \tag{3}$$

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1) All stochastic integrals are defined as integrals in quadratic mean [3].

2) Whenever we take an integral (stochastic or not) of some function, this function is assumed to be integrable.

which means that in any moment t, X contains all statistical information about Y. The process Y, which satisfies (3), is said to be *submitted* to the process X.

Remark 1. Even when the condition (3) is satisfied, we do not know anything about the relations between multiplicities and spectral types of the processes X and Y, [4, 7] — that is why we explicitely supposed that both X and Y belong to C.

Remark 2. The preposition (2) is a little bit unusual, but completely natural, because of the following: if we do not suppose that (2) holds, we can put $Y(t)=\hat{Y}(t)+\tilde{Y}(t)$, $0\leqslant t\leqslant 1$, where $\hat{Y}(t)$ is the mean square estimation of $Y(t)$ by $\{X(s),\ 0\leqslant s\leqslant 1\}$ (then, it follows that $\hat{Y}(t)\in H(X;\, t)$), and $\tilde{Y}(t)$ is the orthogonal error in estimation of $Y(t)$ by $\hat{Y}(t)$. Since the „orthogonal part" $\tilde{Y}(t)$ of $Y(t)$ can not be estimated by present and past of the process X, we can only consider the estimation of $\hat{Y}(t)$ by the elements of the space $H(X;\, t)$. Therefore, the condition (3) is not a restriction, but rather a technical simplification.

2. Let us suppose that the norm of X is a square integrable function, i.e. $\|X(\cdot)\|\in L^2_{[0;\,1]}$ ($L^2_{[0;\,t]}$ is the Hilbert space of square integrable functions on $[0;\,1]$ with respect to the ordinary Lebesgue measure). It is easy to see that the set of all elements of the form

$$\int_0^t f(s)\,X(s)\,ds,\quad f(\cdot)\in L^2_{[0;\,t]}, \tag{4}$$

is dense in $H(X;\, t)$, $0\leqslant t\leqslant 1$.

Really, as the functions $f(\cdot)$ and $\|X(\cdot)\|$ are square integrable, the following inequality holds:

$$\Big\|\int_0^t f(s)\,X(s)\,ds\Big\|^2\leqslant\int_0^t f^2(s)\,ds\cdot\int_0^t\|X(s)\|^2\,ds<\infty;$$

hence the integral (4) exists as an integral in quadratic mean [2], and the element defined by that integral belongs to the space $H(X;\, t)$. If we denote by S_t^X the set of all elements of the form (4), then the relation $S_t^X\subset H(X;\, t)$ holds for all t. It is easy to see that, owing to the mean square continuity of the process X, we have

$$\Big\|X(t)-\int_{t-1/n}^t n\,X(s)\,ds\Big\|\to 0,\ n\to\infty,\ 0<t\leqslant 1,$$

which means that $X(t)$ is the accumulation point of the set S_t^X, i.e. $X(t)\in\overline{S_t^X}$, $0\leqslant t\leqslant 1$. Therefore, arbitrary x from $H(X;\, t)$ belongs to $\overline{S_t^X}$, and then $H(X;\, t)\subset\overline{S_t^X}$ too, which, together with the obvious relation $\overline{S_t^X}\subset H(X;\, t)$, means that the equality $H(X;\, t)=\overline{S_t^X}$ holds for all t. Thus, we proved our assertion.

The previous result implies the following: If Y is submitted to X, then, for every $\varepsilon>0$, there exists a family of functions $\{h_\varepsilon(t, u)$, t-parameter, $u\leqslant t\}$, $h_\varepsilon(t, \cdot)\in L^2_{[0;\,1]}$, so that the process Y_ε, defined by

(5) $$Y_\varepsilon(t)=\int_0^t h_\varepsilon(t, u)\, X(u)\, du, \quad 0\leqslant t\leqslant 1,$$

satisfies the inequality

(6) $$\| Y(t)-Y_\varepsilon(t)\|<\varepsilon, \quad 0\leqslant t\leqslant 1.$$

From (5) it follows that Y_ε is also sumbitted to X.

Remark 3. It is well known that Y_ε can be represented in the form

(7) $$Y_\varepsilon(t)=\int_0^t g_\varepsilon(t, u)\, dz(u), \quad 0\leqslant t\leqslant 1,$$

where z is the innovation process of the process X from (1).

The following equality between $h_\varepsilon(t, \cdot)$, $g_\varepsilon(t, \cdot)$ and $g(t, \cdot)$ (from (1)) obviously holds for all t:

$$g_\varepsilon(t, v)-\int_v^t h_\varepsilon(t, u)\, g(u, v)\, du=0 \text{ a.e. } [\text{mod}\, F_X] \text{ on } [0;\ t].$$

However, the previous equality has not a practical importance since the both families $\{h_\varepsilon(t, \cdot)$ and $\{g_\varepsilon(t, \cdot)\}$ are unknown. The situation is the same if we want to determine one of the representations (5) and (7) of Y_ε, because then we have to use the inequality (6), where the unknown process Y appears. Consequently, we have to find the possibility to determine some other estimation Y_1 of Y, which, perheps, will not satisfy the inequality of the type (6), but could be obtained with less information about Y, or about the relation between X and Y.

3. The process X could be always written in the form $X(t)=Y(t)+x_t$, $0\leqslant t\leqslant 1$, i.e. as the sum of „signal" and „noise", but such decomposition is interesting only when signal and noise are in some special relation, [5]. One of the „useful" relations is the orthogonality of the noise x_t, at every moment t, to the „past" and „present" of the signal Y, i.e. the orthogonality of x_t to all elements $Y(s)$ for $s\leqslant t$. It is clear that, in general, signal and noise do not possess such property, but we shall just see that the process X can be always written as the sum of two processes, one of which is orthogonal to the past and present of the signal Y.

Lemma 1. *For every random process X from C there exist processes Y_1 and V, such that*

(8) $$X(t)=Y_1(t)+V(t), \quad 0\leqslant t\leqslant 1;$$

(9) $$Y_1 \text{ is submitted to } Y;$$

(10) *For all t, $V(t)$ is orthogonal to the past and present of Y.*

P r o o f is obvious. Really, it is easy to see that the processes Y_1 and V, defined by

$$Y_1(t)=P_{H(Y;\,t)}X(t), \quad V(t)=P_{H(X)\ominus H(Y;\,t)}X(t), \quad 0\leqslant t\leqslant 1,$$

satisfy the equality (8) and have the properties (9) and (10).

Let $r_{xx}(\cdot,\,\cdot)$ and $r_{xy}(\cdot,\,\cdot)$ be the correlation function of X and the cross-correlation function between X and Y. The following proposition gives sufficient conditions, in terms of these functions, that the process Y_1 from (8) has the representation of the form (5).

T h e o r e m 1. *The process Y_1 from* (8) *can be represented in the form*

$$Y_1(t)=\int_0^t h_1(t,\,u)\,X(u)\,du, \quad 0\leqslant t\leqslant 1, \tag{11}$$

if the functions $h_1(t,\,\cdot)\in L^2_{[0;\,1]}$, $0\leqslant t\leqslant 1$, satisfy the following conditions:

$$r_{xy}(t,\,s)=\int_0^t r_{xy}(u,\,s)\,h_1(t,\,u)\,du \textit{ for all } t \textit{ and } s\leqslant t; \tag{12}$$

(13) *If, for arbitrary but fixed t, there is a function $f_t(\cdot)\in L^2_{[0;\,1]}$, such that the equality $\int_0^1 r_{xy}(u,\,s)\,f_t(u)\,du=0$ is satisfied for all $s\leqslant t$, then the equality*

$$\int_0^1 f_t(u)\int_0^s r_{xx}(u,\,v)\,h_1(s,\,v)\,dv\,du=0$$

is also satisfied for all $s\leqslant t$.

P r o o f. The condition (12) is equivalent to

$$(X(t),\ Y(s))=\Big(\int_0^t h_1(t,\,u)\,X(u)\,du,\ Y(s)\Big)$$

for any t and all $s\leqslant t$, which means that, for any t, the difference $X(t)-\int_0^t h_1(t,\,u)\,X(u)\,du$ is orthogonal to $H(Y;\,t)$. But, this fact does not imply that $\int_0^t h_1(t,\,u)\,X(u)\,du$ belongs to $H(Y;\,t)$. However, the condition (13) can be written in the following way:

If, for arbitrary but fixed t, there is a function $f_t(\cdot)\in L^2_{[0;\,1]}$ which satisfies the equality $\Big(\int_0^t f_t(u)\,X(u)\,du,\ Y(s)\Big)=0$ for all $s\leqslant t$, then the equality

$\Big(\int_0^t f_t(u)\,X(u)\,du,\ \int_0^t h_1(s,\,v)\,X(v)\,dv\Big)=0$ also holds for all $s\leqslant t$,

which is equivalent to:

If $\int_0^1 f_t(u)\, X(u)\, du$ is orthogonal to $H(Y; t)$, then it is also orthogonal to $H(Y_1; t)$, where Y_1 is defined by (11).

Thus, the condition (13) provides that the element $Y_1(t)$, defined by (11), belongs to the space $H(Y; t)$, which, together with (12), means that $Y_1(t)$ is the projection of $X(t)$ on $H(Y; t)$. The theorem is proved.

It is clear that the conditions (12) and (13) are not necessary for the representation (11) of Y_1 (from (8)) to exist. More precisely, it is possible that there are processes Y_1 and V, which satisfy (8), (9) and (10), and moreover Y_1 has the representation (11), but the family $\{h_1(t, u),\ t\text{-parameter},\ v \leqslant t\}$ satisfies neither (12) nor (13). It is the case when $Y_1(t)$ is not defined as the projection of $X(t)$ onto $H(Y; t)$.

4. We say that Y is *fully submitted* to X if it is submitted to X and if

$$H(Y) \ominus H(Y; t) \subset H(X) \ominus H(X; t), \quad 0 \leqslant t \leqslant 1. \tag{14}$$

Arbitrary subspace $\mathfrak{m}$ of the space $H(X)$ *reduces* the resolution of the identity E_X of X if

$$E_X(s)\, \mathfrak{m} \subset \mathfrak{m}, \quad E_X(s)\, (H(X) \ominus \mathfrak{m}) \subset H(X) \ominus \mathfrak{m}, \quad 0 \leqslant s \leqslant 1.$$

Lemma 2. *The following two statements are equivalent:*

(I) *Y is fully submitted to X.*

(II) *For every t, the space $H(Y; t)$ reduces the resolution of the identity E_X.*

Proof. Let us show that (II) implies (I) (the inverse is obvious). From the facts that the multiplicity of X is equal to one and the spaces $H(Y; t)$ reduce E_X, it follows that the spaces $H(Y; t)$ are cyclic [8]· Thus, there exists an element $\xi \in H(X)$, so that

$$H(Y) = \overline{\mathscr{L}}\{E_X(s)\, \xi, \quad 0 \leqslant s \leqslant 1\}$$

($\overline{\mathscr{L}}$ is the closure of the linear manifold of the elements in the parentheses), and

$$H(Y; t) = \overline{\mathscr{L}}\{E_X(s)\, E_X(t)\, \xi,\ 0 \leqslant s \leqslant 1\} = \overline{\mathscr{L}}(E_X(s)\, \xi, \quad 0 \leqslant s \leqslant t\},$$

which is equivalent to $H(Y; t) \subset H(X; t)$. Also

$$H(Y) \ominus H(Y; t) = \overline{\mathscr{L}}\{(E_X(s) - E_X(t))\, \xi, \quad t < s \leqslant 1\},$$

so that (14) is true. The equivalence of (I) and (II) is proved.

We already said that from the fully submission of Y_1 to X it follows [8] that the space $H(Y_1)$ is cyclic with respect to E_X.

The decomposition (8) becomes very important if processes Y_1 and V are mutually orthogonal. In this case Y_1 (as well as V) is fully submitted to the process X [10].

If the process Y is fully submitted to X, then, obviously, Y_1 is fully submitted to X, and the equality

$$H(Y; t) = H(Y_1; t), \quad 0 \leqslant t \leqslant 1,$$

holds. It is clear that, if Y_1 is fully submitted to X, then Y need not be.

We have just seen that, when processes Y_1 and V from (8) are mutually orthogonal, then Y_1 has the following two characteristic properties: it is submitted to Y, and fully submitted to X. The question arises whether there always exists a non-zero process submitted to Y and fully submitted to X (we suppose that Y is submitted, but not fully submitted to X). Let us suppose that the answer to this question is affirmative and denote by Y_* a process submitted to Y and fully submitted to X. Owing to the previous facts it follows that the space $H(Y_*)$ is cyclic with respect to E_X and $H(Y_*; t) \subset H(Y; t)$ for all t, i.e. there exists $\xi_0 \in H(Y)$ so that

$$E_X(t)\,\xi_0 = E_Y(t)\,\xi_0, \quad 0 \leqslant t \leqslant 1, \tag{15}$$

and

$$H(Y_*; t) = \overline{\mathscr{L}}\{E_Y(s)\,\xi_0, \quad 0 \leqslant s \leqslant t\}, \quad 0 \leqslant t \leqslant 1. \tag{16}$$

It is easy to see that Y_* is also fully submitted to Y. Really, if we define a process Ξ_0 by $\Xi_0(t) = E_Y(t)\,\xi_0$, then

$$H(Y_*; t) = H(\Xi_0; t) \subset H(Y; t)$$

for all t, and since Ξ_0 is the orthogonal increments process, we have

$$H(\Xi_0) \ominus H(\Xi_0; t) = \overline{\mathscr{L}}\{\Xi_0(s) - \Xi_0(t), \quad t < s \leqslant 1\} \subset$$
$$\subset H(Y) \ominus H(Y; t), \quad 0 \leqslant t \leqslant 1.$$

Therefore, the space $H(Y; t)$ could be written as the orthogonal summ

$$H(Y; t) = H(\Xi_0; t) \oplus H_{1,t}, \quad 0 \leqslant t \leqslant 1. \tag{17}$$

The spaces $H_{1,t}$ do not reduce the resolution of the identity E_X (really, if $H_{1,t}$ reduces E_X, then $H(Y; t)$ from (17) reduces E_X, which is, as we already saw, equivalent to the fact that Y is fully submitted to X, and this contradicts our assumption); it is clear that the relations

$$H_{1,t} \subset H(Y; t) \subset H(X; t)$$

hold for all t.

Let Y_1 be a process which satisfies the equality

$$H(Y_1; t) = H_{1,t}, \quad 0 \leqslant t \leqslant 1.$$

According to our assumption, there exists a process Ξ_1, fully submitted to X and submitted to Y_1 so that the following equatity is true:

$$H(Y; t) = H(\Xi_0; t) \oplus H(\Xi_1; t) \oplus H_{2,t}, \quad 0 \leqslant t \leqslant 1.$$

If we continue the described procedure in infinity, we shall obtain (because of the separability of the space $H(Y)$) the equality

$$H(Y; t) = \sum_{i=0}^{\infty} \oplus H(\Xi_i; t), \quad 0 \leqslant t \leqslant 1,$$

where every process Ξ_i is fully submitted to X. But, that means that Y itself is fully submitted to X, contrary to our assumption.

Consequently, we proved that if Y is submitted to X, then process Y_*, submitted to Y and fully submitted to X, need not exist.

5. The problem how to determine conditions for the process Y_* to exist is an interesting one. In the next proposition we shall give an solution of this problem in the case when the generating element λ of the space $H(X)$ can be represented in the form

$$\lambda=\int_0^1 l(u)\,X(u)\,du, \tag{18}$$

where $l(u)\in L^2_{[0;\,1]}$.

Theorem 2. *Suppose that the process X belongs to C, and that the generating element λ of $H(X)$ can be represented in the form* (18). *Suppose also that Y is arbitrary process submitted to X. The process Y_*, submitted to Y and fully submitted to X, exists if and only if there exists a function $\chi(\cdot)$, with values 0 and 1, measurable with respect to F_X, such that, for any t, the element*

$$\int_0^t \chi(u)\,dE_X(u)\,\lambda \tag{19}$$

can be approximated arbitrarily well by elements of the form

$$\int_0^t k_t(u)\,Y(u)\,du,\quad k_t(\cdot)\in L^2_{[0;\,t]},$$

or, equivalenty, such that there exist elements

$$\lambda_{t,n}=\int_0^t k_{t,n}(u)\,Y(u)\,du,\quad k_{t,n}(\cdot)\in L^2_{[0;\,t]},\quad n=1,\ 2, \tag{20}$$

for which

$$\left\|\int_0^t \chi(u)\,dE_X(u)\,\lambda-\lambda_{t,n}\right\|\to 0,\ n\to\infty,\ 0\leqslant t\leqslant 1. \tag{21}$$

Proof. If the process Y_* exists, then [6, 8] a generating element of $H(Y_*)$ can be represented in the form $\int_0^1 \chi(u)\,dE_X(u)\,\lambda$, where $\chi(\cdot)$ has the above properties, and element λ, defined by (18), generates the space $H(X)$. As element (19) belongs to $H(Y;t)$, then, according to the result of the section 2, it can be approximated arbitrarily well by the elements of the form (20), i.e. (21) holds.

Inversely, if the relations (18)—(21) hold, this means that for any t the element defined by (19) belongs to $H(Y;t)$ (as a limit of the elements (20), which belong to this space). Thus, if we put

$$Y_*(t)=\int_0^t \chi(u)\,dE_X(u)\,\lambda,\quad 0\leqslant t\leqslant 1,$$

then $Y_*(t) \in H(Y; t)$ for all t. Besides, $H(Y_*)$ is cyclic with respect to E_X, and generated by the element $\int_0^1 \chi(u)\, dE_X(u)\, \lambda$, [8], which means that Y_* is fully submitted to X. The proof is completed.

Remark 4. The element λ of the form (18) will be a generating element of $H(X)$ if and only if λ induces the spectral type which is equivalent to the spectral type E_X [8]. Obviously this would be the case if

$$\int_v^1 l(u)\, g(u, v)\, du \neq 0 \ \text{ a.e. } [\bmod F_X],$$

where the functions $g(\cdot, \cdot)$ are from (1).

Remark 5. If X is the orthogonal increments process and $l(u) > 0$ a.e. $[\bmod F_X]$ on $[0; 1]$, then the element $\lambda = \int_0^1 l(u)\, X(u)\, du$ generates spectral type equivalent to F_X. Really, it is easy to see that λ generates the spectral type

$$d\Phi_\lambda(t) = \Big[\int_t^1 l(u)\, du\Big]^2 dF_X(t), \quad 0 \leqslant t \leqslant 1,$$

which means that the Radon-Nikodym derivative of the measure Φ_λ with respect to F_X is positive almost everywhere $[\bmod F_X]$.

Remark 6. Let X be a Markov process in the wide sense [6]. The generating element λ of X can be represented in the form

$$\lambda = \int_0^1 l(u) \frac{1}{g(u)} X(u)\, du,$$

where $l(u) > 0$ for all u, and the function $g(\cdot)$ is defined by

$$g(t) = \begin{cases} \dfrac{1}{a(s_0, t)}, & 0 \leqslant t \leqslant s_0, \\ a(t, s_0), & s_0 < t \leqslant 1, \end{cases}$$

s_0 is an arbitrary point from $[0; 1]$, $a(t, s) = \dfrac{r(t, s)}{r(s, s)}$, $s \leqslant t$, and $r(\cdot, \cdot)$ is the correlation function of X.

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