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# ON SOME CLASSES OF UNRECOGNIZABLE PROPERTIES OF GROUPS

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**Introduction.** The algorythmic solvability of problems in group theory has attracted much attention ([1] — [8],, [12] — [16]). For a number of properties of the elements of a given group, as well as for properties of a group as a whole, it is already proved to be algorythmicaly (i.e. recursively) unrecognizable. In this paper we expound some further results in this direction.

The groups are supposed to be finitely presentable (f.p.), i.e. to have a presentation $\Pi = \langle x_1, \ldots, x_n; R_1 = 1, \ldots, R_k = 1 \rangle$ where $x_1, \ldots, x_n$ are generators and $R_1 = 1, \ldots, R_k = 1$ the defining relations. The group $G_\Pi$ defined by this presentation is $G_\Pi \cong F_n / [R_1, \ldots, R_k]$, $F_n$ being a free group of rank $n$, and $[R_1, \ldots, R_k]$ the minimal normal subgroup of $F_n$ containing the words $R_1, \ldots, R_k$.

The properties are supposed to be algebraic, i.e. preserved under isomorphisms.

The problems we are dealing with are of the following type: given an algebraic property $P$ of f.p. groups, prove or disprove the existence of an algorythm to decide for any given finite presentation $\Pi$ whether $P(G_\Pi)$ holds or not.

**Statements and proofs.** Let $\mathcal{U}$ denote a class of all universal f.p. groups, i.e. groups which contain as a subgroup an isomorphic copy of every f.p. group (the existence of such groups was proved by G. Higman [9]). $\mathcal{M}(P)$ denotes that a nottrivial algebraic property $P$ of f.p. groups is a Markov property, i.e. that there exists a group $F$ which cannot be embedded into a group $G$ such that $P(G)$ is true.

In what follows we shall make use of the following results.

T h e o r e m (Rabin [15], Adjan [1]); Every Markov property of f.p. groups is not recursively recognizable.

Theorem (Božović [5]): *A nontrivial algebraic property $P$ of f.p. groups is a Markov property if and only if none of the universal f.p. groups enjoys $P$, i.e.*

$$\mathcal{M}(P) \Leftrightarrow (\forall U \in \mathcal{U}) \neg P(U).$$

Hence, one should search for possibly recognizable ones only among those properties that contain both some (but not all) of the universal groups, and some of the non-universal ones. Even some properties of this kind are already known to be unrecognizable: some of the strong Markov properties [8], being a Hopf group [7], the properties incompatible with free product [11] etc. As for the remaining properties, it is still open which of them are recognizable and which are not; by the results given here we resolve this problem for some of them.

Let $f(G)$ denote a maximal number of nontrivial factors in free decomposition of the group $G$, i.e.

$$f(G)=k \Leftrightarrow (\exists G_1, \ldots, G_k)\, G \cong G_1 * \cdots * G_k \;\wedge$$
$$(\forall m \in N)\, ((\exists H_1, \ldots, H_m)\, G \cong H_1 * \cdots * H_m \Rightarrow m \leqslant k)$$

In what follows we are dealing with algebraic properties $P$ of the form $P=Q \cup R$, where $Q$ contains only some (but not all) of the universal groups, and $R$ contains only some of the non-universal ones.

Result 1: *Let $P=Q \cup R$ be an algebraic property of f.p. groups. If there exists a positive integer $k$ such that*

(i) $(\forall G \in R)\, f(G) \neq k \wedge \min\limits_{G \in R} f(G) > 1$, *or*

(ii) $(\forall G \in Q)\, f(G) \neq k$,

*then $P$ is a recursively unrecognizable property.*

Proof: Let $\Pi$ be a presentation of an f.p. group with unsolvable word problem and let $f: (\Pi, r) \rightarrow \Pi(r)$ be a recursive function which assigns to the ordered pair consisting of a presentation $\Pi$ and a word $r$, $r \in \Pi$, another presentation $\Pi(r)$, satisfying

$$\vdash_{\Pi} r=1 \Rightarrow G_{\Pi(r)} \cong 1$$

$$\vdash_{\Pi} r \neq 1 \Rightarrow G_{\Pi} < G_{\Pi(r)} \qquad \text{(M. Rabin, [15]).}$$

Let us consider first the case (i) when $R$ fulfills the required condition, i.e. when a positive integer $k$ $(k>1)$ exists such that $R$ does not contain a group having $k$ as the maximal number of factors in its free decomposition. For the sake of simplicity we choose the smallest such $k$.

Let $N$ be a (non-universal) group enjoying $R$ i.e. $P$, and let $f(N)=k-1$. For $\Pi$ we choose a presentation of a torsion-free group (i.e. one that has no element of finite order) with unsolvable word problem. (W. Boone [4] proved that, for every recursively enumerable degree $d$, there exists a torsion-free f.p. group with word problem of degree $d$.) Let $r$ be a word on the generators of $\Pi$, and $\Pi(r)$ a presentation described above.

Now, let us consider the following group $G_r$

(1) $$G_r = N * (G_{\Pi(r)} \times G_{\Pi(r)}).$$

Then, one has

$$\underset{\Pi}{\vdash} r = 1 \;\Rightarrow\; G_{\Pi(r)} \cong 1 \;\Rightarrow\; G_r \cong N \;\Rightarrow\; P(G_r).$$

If $\underset{\Pi}{\vdash} r \neq 1$, then $G_\Pi < G_{\Pi(r)}$ and $G_r$ is of the form (1). If $G_\Pi$ is a torsion-free group, the same applies to $G_{\Pi(r)}$ (W. Boone [4]), and hence $G_{\Pi(r)}$ is not a universal group (because a universal group contains an isomorphic copy of every finite group, and hence contains elements of every finite order). As the direct product of two torsion-free groups is a torsion-free group also, $G_{\Pi(r)} \times G_{\Pi(r)}$ is not a universal group. So, $G_r$ is not a universal group too (N. Božović [6]), and $f(G_r) = k$ (as $G_{\Pi(r)} \times G_{\Pi(r)}$ cannot be decomposed into a nontrivial free product, A. G. Kurosh [10]). So, $G_r$ does not enjoy $P$.
Consequently, $G_r$ enjoys $P$ iff $G_{\Pi(r)} \cong 1$.

In the case (ii), when it is $Q$ which satisfies the above mentioned condition, the proof goes analogously. Here, in the case $k > 1$, the group $G_r$ can be taken as

$$G_r = U * (G_{\Pi(r)} \times G_{\Pi(r)})$$

where $U$ is a universal f.p. group enjoying $P$ and having $k-1$ as the maximal number of free factors.

If $k = 1$, we choose

$$G_r = U \times (G_{\Pi(r)} \times G_{\Pi(r)})$$

where $U$ has property $P$.
So, $G_r$ enjoys $P$ iff $G_{\Pi(r)} \cong 1$, too.

Now one can proceed as usual: if $P$ is an algorythmicaly recognizable property over the class of all finite presentations, then for every given $r$, $r \in \Pi$ (where $\Pi$ is chosen as above), one can construct effectively the group $G_r$ and check whether $P(G_r)$ is true or false. Thus one can decide for every $r$, $r \in \Pi$, whether $\underset{\Pi}{\vdash} r = 1$ or not, contrary to the assumption that $\Pi$ has unsolvable word problem.

Result 2: *Let $P = Q \cup R$ be an algebraic property of f.p. groups, and let $d$ be an arbitrary given degree of unsolvability. If $R$ does not contain a group with word problem of degree greater then or equal to $d$, then $P$ is a recursively unrecognizable property.*

Proof: The proof is similar to the proof of Result 1. Here we choose

$$G_r = G_1 * G_{\Pi(r)}$$

where $G_1$ is the group such that $P(G_1)$ holds, and $\Pi$ is the presentation of the torsion-free group $G_\Pi$ such that $dg\,(G_\Pi) \geqslant d$. By Rabin's construction [15], $G_{\Pi(r)} > G_\Pi$, so that

$$dg\,(G_{\Pi(r)}) \geqslant dg\,(G_\Pi).$$

Consequently, $P(G_r)$ iff $G_{\Pi(r)} \cong 1$, etc.

Discussion. It is known that many group properties are not algorythmicaly recognizable, as mentioned in the Introduction. However, the exact border line, dividing unrecognizable ones from the rest, is still unknown.

Using the Results 1 and 2 given above, we can restrict our search for the algorythmicaly recognizable properties $P$ to those for which it is true that:

(1) for every $n \in N$, $(n > 1)$, $P$ contains a group $G$ with $f(G) = n$,

(2) for every recursively enumerable degree of unsolvability $d$, $P$ contains a group with the word problem of degree $d$,

(3) both (1) and (2) are true for $\neg P$ also.

It is possible that some relevant group properties still fall within this class.

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