PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 26 (40), 1979, pp. 5–9

RESULTS ON NON-UNIQUE FIXED POINTS*

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(Received May 16, 1978)

1. Introduction:

Recently Ćirić [2] proved some fixed point theorems when the mapping T of a metric space (M, d) satisfies an inequality of the type

$$(1)\quad \min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \leqslant \alpha\, d(x, y)$$

for $x, y \in M$ and $0 < \alpha < 1$. He also showed that the mapping should be orbitally continuous. In a recent paper we [1] have further extended this idea of Ćirić.

The chief aim of the present paper is to introduce and to study the fixed points of a mapping T by using symmetrical rational expression and which satisfies the inequality

$$\frac{\min\{d(Tx,Ty)\,d(x,y),\, d(x,Tx)\,d(y,Ty)\} - \min\{d(x,Tx)\,d(x,Ty),\, d(y,Ty)\,d(y,Tx)\}}{\min\{d(x\ Tx),\, d(y,\ Ty)\}} \leqslant$$

$$(2)\qquad \leqslant \alpha\, d(x, y)$$

for $x, y \in M$, $0 < \alpha < 1$ and $d(x, Tx) \neq 0$, $d(y, Ty) \neq 0$.

Localized version of the main theorem and sequence of mappings which satisfy (2) are also discussed.

Before going into the theorems we state the following definitions.

Definition 1 (Ćirić [3]). A mapping T on a metric space M is orbitally continuous if $\lim_i T^{n_i} x = u$ implies $\lim_i T T^{n_i} x = Tu$ for each $x \in M$.

Definition 2 (Ćirić [3]). A space M is T-orbitally complete if every Cauchy sequence of the form $\{T^{n_i} x\}_{i=1}^{\infty}$, $x \in M$ converges in M.

* Research supported by C.S.I R.

2. Fixed point theorems.

Theorem 1. *Let M be T-orbitally complete metric space and $T: M \to M$ be an orbitally continuous mapping on M. If T satisfies the following condition*

$$\frac{\min\{d(Tx, Ty)\, d(x, y),\, d(x, Tx)\, d(y, Ty)\} - \min\{d(x, Tx)\, d(x, Ty),\, d(y, Ty)\, d(y, Tx)\}}{\min\{d(x, Tx),\, d(y, Ty)\}} \leqslant$$

$$\leqslant \alpha\, d(x, y) \tag{3}$$

for some $0<\alpha<1$, $x, y \in M$, $d(x, Tx) \neq 0$, $d(y, Ty) \neq 0$. Then for each $x \in M$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of T.

Proof. We define a sequence of elements $\{x_n\}$ of M as follows: Let x_0 be any element of M. Let

$$x_1 = T x_0, \quad x_2 = Tx = Tx_1, \ \ldots \ x_n = Tx_{n-1}, \ \ldots$$

Now

$$\frac{\min\{d(Tx_0, Tx_1)\, d(x_0, x_1),\, d(x_0, Tx_0)\, d(x_1, Tx_1)\} - \min\{d(x_0, Tx_0)\, d(x_0, Tx_1)\, d(x_1, Tx_1)\, d(x_1, Tx_0)\}}{\min\{d(x_0, Tx_0),\, d(x_1, Tx_1)\}}$$

$$\leqslant \alpha\, d(x_0, x_1)$$

i. e.
$$\frac{\min\{d(x_1, x_2)\, d(x_0, x_1),\ d(x_0, x_1)\, d(x_1, x_2)\}}{\min\{d(x_0, x_1),\ d(x_1, x_2)\}} \leqslant \alpha\, d(x_0, x_1)$$

i. e.
$$\frac{d(x_1, x_2)\, d(x_0, x_1)}{\min\{d(x_0, x_1),\ d(x_1, x_2)\}} \leqslant \alpha\, d(x_0, x_1)$$

Now if $d(x_1, x_0)$ is minimum, then we get

$$d(x_1, x_2) \leqslant \alpha\, d(x_0, x_1)$$

and if $d(x_1, x_2)$ is minimum then we have

$$d(x_0, x_1) \leqslant \alpha\, d(x_0, x_1)$$

which is a contradiction since $\alpha<1$.
So we get

$$d(x_1, x_2) \leqslant \alpha\, d(x_0, x_1)$$

Proceeding in this manner we obtain

$$d(x_n, x_{n+1}) \leqslant \alpha\, d(x_{n-1}, x_n) \leqslant \alpha^2 d(x_{n-2}, x_{n-1}) \leqslant \cdots \leqslant \alpha^n d(x_0, x_1)$$

Hence for any integer p one has

$$d(x_n, x_{n+p}) \leqslant \sum_{j=n}^{n+p-1} d(x_j, x_{j+1}) \leqslant \left(\sum_{j=n}^{n+p-1} \alpha^j\right) d(x_0, x_1) \leqslant \frac{\alpha^n}{1-\alpha}\, d(x_0, x_1)$$

Hence it follows that the sequence $\{x_n\}$ is Cauchy. M being T-orbitally complete, there is some $u \in M$ such that $u = \lim_n T^n x$. By orbital continuity of T

$$Tu = \lim_n T T^n x = u$$

and hence u is a fixed point of T.

Theorem 2. *Let*

$$A=A(x_0, r)=\{x\in M \mid d(x_0, x)\leqslant r\}$$

where (M, d) *is an orbitally complete metric space. Let* T *be an orbitally continuous mapping of* A *into* M *and satisfies*

(3) *for* $x, y\in A$ *and*

(4) $$d(x_0, Tx_0)\leqslant(1-\alpha)r$$

Then T *has a fixed point.*

Proof. By (4) we have

$$x_1=Tx_0\in A(x_0, r)$$

and by (2)

$$\frac{\min\{d(Tx_0, Tx_1)\, d(x_0, x_1), d(x_0, Tx_0)\, d(x_1, Tx_1)\}-\min\{d(x_0, Tx_0)\, d(x_0, Tx_1), d(x_1, Tx_1)\, d(x_1, Tx_0)}{\min\{d(x_0, Tx_0), d(x_1, Tx)\}}$$

$$\leqslant\alpha\, d(x_0, x_1)$$

Following lines of arguments of Theorem 1 we get

$$d(x_1, x_2)\leqslant\alpha\, d(x_0, x_1)\leqslant\alpha(1-\alpha)r$$

Hence

$$\begin{aligned} d(x_0, x_2)&\leqslant d(x_0, x_1)+d(x_1, x_2) \qquad [x_2=Tx_1]\\ &\leqslant(1-\alpha)r+\alpha(1-\alpha)r=(1+\alpha)(1-\alpha)r. \end{aligned}$$

Suppose that

$$d(x_0, x_n)\leqslant(1+\alpha+\cdots+\alpha^{n-1})(1-\alpha)r$$

and that

$$d(x_{n-1}, x_n)\leqslant\alpha^{n-1}(1-\alpha)r \qquad [x_n=Tx_{n-1}]$$

Then

$$\frac{\min\{d(x_n, Tx_n)\, d(x_{n-1}, x_n), d(x_n, x_{n-1}) d(x_n, Tx_n)\}-\min\{d(x_n, x_{n-1})\, d(x_{n-1}, Tx_n), d(x_n, Tx_n) d(x_n, x_n)\}}{\min\{d x_n, x_{n-1}), d(x_n, Tx_n)\}}$$

$$\leqslant\alpha\, d(x_{n-1}, x_n)$$

That is

$$d(x_n, Tx_n)\leqslant\alpha\, d(x_{n-1}, x_n)\leqslant\alpha^n(1-\alpha)r$$

Therefore

$$\begin{aligned} d(x_0, x_{n+1})&\leqslant d(x_0, x_n)+d(x_n, x_{n+1})\\ &\leqslant(1+\alpha+\cdots+\alpha^{n-1})(1-\alpha)r+\alpha^n(1-\alpha)r\\ &\leqslant(1+\alpha+\cdots+\alpha^n)(1-\alpha)r\leqslant r \end{aligned}$$

Thus the sequence x_0, $x_{n+1}=Tx_n$, $n\geqslant 0$ is contained in A. Also

$$\begin{aligned} d(x_n, x_m)&\leqslant d(x_n, x_{n+1})+\cdots+d(x_{m-1}, x_m)\\ &\leqslant(\alpha^n+\cdots+\alpha^{m-1})(1-\alpha)r\leqslant\alpha^n r\to 0 \end{aligned}$$

as $n\to\infty$. Since A is orbitally complete, so $u=\lim T^n x$ for some $u\in A$. By orbital continuity of T we have

$$Tu=\lim_n TT^n x=u$$

Thus u is a fixed point of T and hence the proof.

3. Sequence of mappings.

Theorem 3. *Let (M, d) be a compact metric space. $\{T_n\}$ be a sequence of orbitally continuous functions of M into itself and satisfies*

$$\frac{\begin{aligned}&\min\{d(T_n x, T_{n+1}y)\,d(x, y),\ d(x, T_n x)\,d(y, T_{n+1}y)\}-\\ &-\min\{d(x, T_n x)\,d(x, T_{n+1}y),\ d(y, T_{n+1}y)\,d(y, T_n x)\}\end{aligned}}{\min\{d(x, T_n x),\ d(y, T_{n+1}y)\}}\leqslant\alpha\, d(x, y) \tag{5}$$

for some $0<\alpha<1$ and all $x, y\in M$, $d(x, T_n x)\neq 0$, $d(y, T_{n+1}y)\neq 0$ and each $n=1, 2, \ldots$. If $\{T_n\}$ converges pointwise to the function T, then T has a fixed point. Indeed every cluster point of a sequence $\{x_n\}$ of fixed points x_n of T_n is a fixed point of T.

Proof. Let $x\in M$ be aribitrary and construct and construct the sequence

$$x_0=x,\quad x_1=T_1 x_0,\quad x_2=T_2 x_1,\ \ldots\ x_n=T_n x_{n-1},\ \ldots$$

By (5) we have

$$\frac{\begin{aligned}&\min\{d(T_n x_{n-1}, T_{n+1}x_n)\,d(x_n, x_{n-1}),\ d(x_{n-1}, T_n x_{n-1})\,d(x_n, T_{n+1}x_n)\}-\\ &-\min\{d(x_{n-1}, T_n x_{n-1})\,d(x_{n-1}, T_{n+1}x_n\,d(x_n, T_{n+1}x_n)\,d(x_n, T_n x_{n-1})\}\end{aligned}}{\min\{d(x_{n-1}, T_n x_{n-1})\ d(x_n, T_{n+1}x_n)\}}\leqslant$$

$$\leqslant\alpha\, d(x_{n-1}\, x_n)$$

which implies

$$d(x_{n+1}, x_n)\leqslant\alpha\, d(x_n, x_{n-1})$$

Proceeding in this way one gets

$$d(x_{n+1}, x_n)\leqslant\alpha\, d(x_n, x_{n-1})\leqslant\cdots\leqslant\alpha^n d(x_0, x_1)$$

By routine calculation one can show that the following inequalities hold

$$d(x_i, x_j)\leqslant\sum_{k=i}^{j-1} d(x_k, x_{k+1})\leqslant\frac{\alpha^i d(x_0, x_1)}{1-\alpha}\ j>i$$

Thus the sequence $\{x_n\}$ is Cauchy. Since M is complete and $\{T_n\}$ are orbitally continuous, T_n has a fixed point for infinitely many of n's. So there is a subsequence $\{T_{n(k)}\}$ of $\{T_n\}$ such that each $T_{n(k)}$ has a fixed point, say x_k.

By compactness we may (by taking a subsequence) assume that $\{x_k\}$ converges to some x in M. We shall show that x is a fixed point of T. If $x_k \neq x$ for only finitely many of k's, then

$$\begin{aligned} Tx &= \lim_{k\to\infty} T_{n(k)}\, x \\ &= \lim_{k\to\infty} T_{n(k)}\, x_k \\ &= \lim_{k\to\infty} x_k \\ &= x \end{aligned}$$

So we may assume that $x_k \neq x$ for infinitely many of k's. This completes the proof of the theorem.

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