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INTERPOLATION FORMULAS OVER FINITE SETS

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Gr. C. Moisil [3] has given new interpolation formulas for switching functions, making use of Sheffer functions alone or in combination with disjunction or conjunction. E. L. Post [4] (cf. R. Wille [5]) and independently M. Carvallo [1] have found a very general iaterpolation formula for functions of the form $f: E^n \to E$, where E is a finite set. The Post-Carvallo theorem generalizes several known results and provides a tool for obtaining new interpolation formulas in various concrete cases; yet it does not include Moisil's formulas. The aim of this note is to point out that Moisil's ideas yield naturally a common generalization of the above mentioned results.

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Let $E_1, \ldots, E_n$ be n finite nonempty sets (not necessarily pairwise distinct) and $p = |E_1 \times \cdots \times E_n|$. Let further E be a set containing two distinguished elements ω and ε, with $\omega \neq \varepsilon$, and endowed with a unary operation$^* : E \to E$, a partially defined p-ary operation $\circ : E^p \to E$ and a partially defined $(n+1)$-ary $\circ : E^{n+1} \xrightarrow{\circ} E$. Whenever the values $\circ(y_1, \ldots, y_p)$ and $\circ(z_0, z_1, \ldots, z_n)$ are defined, they will be denoted by $y_1 \circ y_2 \cdot \cdots \cdot y_p$ and $z_0 \cdot z_1 \cdot \cdots \cdot z_n$, respectively. Finally suppose the given operations fulfil

$$(1) \qquad \begin{gathered}(y_1 \cdot \beta_{11} \cdot \cdots \cdot \beta_{1n}) \cdot \cdots \cdot (y_{i-1} \cdot \beta_{i-1,1} \cdot \cdots \cdot \beta_{i-1,n}) \circ \\ \circ (y_i^* \cdot \varepsilon \cdot \cdots \cdot \varepsilon) \cdot (y_{i+1} \cdot \beta_{i+1,1} \cdot \cdots \cdot \beta_{i+1,n}) \circ \cdots \\ \cdots \circ (y_p \cdot \beta_{p1} \cdot \cdots \cdot \beta_{pn}) = y_i\end{gathered}$$

for every $y_1, \ldots, y_p \in E$, every $\beta_{hk} \in \{\omega, \varepsilon\}$ $(h = 1, \ldots, i-1, i+1, \ldots, p;$ $k = 1, \ldots, n)$ such that $\beta_{hk(h)} = \omega$ for every $h = 1, \ldots, i-1, i+1, \ldots, p$ and some $k(h) \in \{1, \ldots, n\}$, and for every $i = 1, \ldots, p$. The exact meaning

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of this axiom is that the elements $(y_h \cdot \beta_{hl} \cdot \cdots \cdot \beta_{hn})$ $(h=1, \ldots, i-1, i+1, \ldots, p)$, $(y_i^* \cdot \varepsilon \cdot \cdots \cdot \varepsilon)$, as well as the left side of (1), exist and the equality holds.

Theorem. *Under the above assumptions every function*

$$f: E_1 \times \cdots \times E_n \rightarrow E \tag{2}$$

fulfils the identity

$$f(x, \ldots, x_n) = \bigcirc_{\alpha_1 \in E_1, \ldots, \alpha_n \in E_n} [(f(\alpha_1, \ldots, \alpha_n))^* \cdot x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}], \tag{3}$$

where the right side stands for the result of the operation $\circ$ *applied to the* p *elements of the form indicated within the brackets, while for every* $x, \alpha \in E_k$ $(k=1, \ldots, n)$, x^α *is defined by*

$$x^\alpha = \begin{cases} \varepsilon \text{ if } x = \alpha, \\ \omega \text{ if } x \neq \alpha. \end{cases} \tag{4}$$

Proof. Take an arbitrary but fixed vector $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$. Then

$$(f(x_1, \ldots, x_n))^* \cdot x_1^{x_1} \cdot \cdots \cdot x_n^{x_n} = (f(x_1, \ldots, x_n))^* \cdot \varepsilon \cdot \cdots \cdot \varepsilon,$$

while for every $(\alpha_1, \ldots, \alpha_n) \neq (x_1, \ldots, x_n)$ there is an index k such that $\alpha_k \neq x_k$, hence $x_k^{\alpha_k} = \omega$; on the other hand $x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} \in \{\omega, \varepsilon\}$. Now (3) follows from (1).

Remark. The operation $\circ$ (as well as $\cdot$) is not assumed to be symmetric i.e., if $(i_1, \ldots, i_p)$ is a permutation of $(1, \ldots, p)$, the elements $y_1 \circ \cdots \circ y_p$ and $y_{i_1} \circ \cdots \circ y_{i_p}$ need not be equal; as a matter of fact, they need not even exist simultaneously. Yet the equality (3) has been established independently of the $p!$ possible orderings of the vectors $(\alpha_1, \ldots, \alpha_n)$ from $E_1 \times \cdots \times E_n$.

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As a first example, take $E_1 = \cdots = E_n = E\{0, 1\}$; here $p = 2^n$. For each integer $q \geqslant 2$ Gr. C. Moisil [3] has defined the q-ary operations $\wedge, \vee, \top, \perp$ (conjunction, disjunction and the Sheffer functions) as follows:

$$x_1 \wedge \cdots \wedge x_q = \begin{cases} 1 \text{ if } x_1 = \cdots = x_q = 1, \\ 0 \text{ otherwise;} \end{cases}$$

$$x_1 \vee \cdots \vee x_q = \begin{cases} 0 \text{ if } x_1 = \cdots = x_q = 0, \\ 1 \text{ otherwise;} \end{cases}$$

$$x_1 \top \cdots \top x_q = \begin{cases} 1 \text{ if } x_1 = \cdots = x_q = 0, \\ 0 \text{ otherwise} \end{cases}$$

$$x_1 \perp \cdots \perp x_q = \begin{cases} 0 \text{ if } x_1 = \cdots = x_q = 1, \\ 1 \text{ otherwise;} \end{cases}$$

thus $x_1 \top \cdots \top x_q = x_1' \wedge \cdots \wedge x_q'$ and $x_1 \perp \cdots \perp x_q = x_1' \vee \cdots \vee x_q'$, where $'$ stands for negation, i.e., $0'=1$ and $1'=0$. Now the eight interpolation formulas given by Moisil (I—VIII in [3], pp. 223—226) can be obtained from the above Theorem as shown in the following table:

No.	I	II	III	IV	V	VI	VII	VIII
$\circ$	$\vee$	$\wedge$	$\vee$	$\wedge$	$\perp$	$\perp$	$\top$	$\top$
$\bullet$	$\wedge$	$\vee$	$\top$	$\perp$	$\perp$	$\vee$	$\top$	$\wedge$
y^*	y	y	y'	y'	y	y'	y	y'
ω	0	1	1	0	0	1	1	0
ε	1	0	0	1	1	0	0	1

For in cases I, II, VI and VIII we have $y \bullet \beta_1 \bullet \cdots \bullet \omega \bullet \cdots \bullet \beta_n = \omega$, $z \bullet \varepsilon \bullet \cdots \bullet \varepsilon = z$ and $\omega \circ \cdots \circ \omega \circ z \circ \omega \circ \cdots \circ \omega = z^*$, therefore (1) follows taking $z = y_i^*$, whereas in cases III, IV, V and VII we have $y \bullet \beta_1 \bullet \cdots \bullet \omega \bullet \cdots \bullet \beta_n = \varepsilon$, $z \bullet \varepsilon \bullet \cdots \bullet \varepsilon = z'$ and $\varepsilon \circ \cdots \circ \varepsilon \circ z \circ \varepsilon \circ \cdots \circ \varepsilon = (z^*)'$, hence (1) follows from

$$\varepsilon \circ \cdots \circ \varepsilon \circ (y_i^* \bullet \varepsilon \bullet \cdots \bullet \varepsilon) \circ \varepsilon \circ \cdots \circ \varepsilon =$$

$$= \varepsilon \circ \cdots \circ \varepsilon \circ (y_i^*)' \circ \varepsilon \circ \cdots \circ \varepsilon = (((y_i^*)')^*)' = y_i.$$

In cases I, IV, V and VIII we have $\omega = 0$ and $\varepsilon = 1$, so that the functions x^α coincide with Moisil's functions $L_\alpha(x)$ and the reduction of formula (3) to the corresponding Moisil formulas is straightforward. In cases II, III, VI and VII we have $\omega = 1$ and $\varepsilon = 0$, so that $x^\alpha = L_{\alpha'}(x)$ and the reduction of (3) to the corresponding formulas given by Moisil follows by an easy transformation which we illustrate in case III:

$$f(x_1, \ldots, x_n) = \bigvee_{(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n} [f(\alpha_1, \ldots, \alpha_n))' \bullet x_1^{\alpha_1} \bullet \cdots \bullet x_n^{\alpha_n}] =$$

$$= \bigvee_{\alpha_1, \ldots, \alpha_n \in \{0,1\}} [(f(\alpha_1, \ldots, \alpha_n))' \bullet L_{\alpha_1'}(x_1) \bullet \cdots \bullet L_{\alpha_n'}(x_n))] =$$

$$= \bigvee_{\alpha_1, \ldots, \alpha_n \in \{0,1\}} [(f(\alpha_1', \ldots, \alpha_n'))' \bullet L_{\alpha_1}(x_1) \bullet \cdots \bullet L_{\alpha_n}(x_n))].$$

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As a second example, we shall obtain a slight generalization of the Post-Carvalo formula by specialization of the above main Theorem.

Let again E be set containing two distinguished elements ω and ε, with $\omega \neq \varepsilon$. Now suppose E is endowed with two partially defined binary operations $\circ, \bullet : E^2 \to E$ satisfying

(4) $$y \circ \omega = \omega \circ y = y,$$

(5) $$y \bullet \omega = \omega \text{ and } y \bullet \varepsilon = y,$$

for every $y \in E$. Define a collection of partial operations $\circ_q$, $\circ_q : E^q \xrightarrow{\circ} E$ ($q=2, 3, 4, \dots$) as follows: $\circ_2 = \circ$, $\bullet_2 = \bullet$ and

$$y_1 \circ_q \cdots \circ_q y_{-1} \circ_q y_q = (y_1 \circ_{q-1} \cdots \circ_{q-1} y_{q-1}) \circ y_q, \tag{6}$$

$$y_1 \bullet_q \cdots \bullet_q y_{q-1} \bullet_q y_q = (y_1 \bullet_{q-1} \cdots \bullet_{q-1} y_{q-1}) \bullet y_q, \tag{7}$$

for $q=3, 4, \dots$. The exact meaning of either recursive equation is that whenever the elements in the right side exist, the left side exists too and the equality holds. The subscripts q will be omitted in the sequel.

P r o p o s i t i o n. *Under the above assumptions and notation,* (4)—(7), *for every* $n=1, 2, 3, \dots$ *and every* n *finite nonempty sets* $E_1, \dots, E_n$, *each function*

$$f : E_1 \times \cdots \times E_n \to E \tag{2}$$

fulfils the identity

$$f(x_1, \dots, x_n) = \bigcirc_{\alpha_1 \in E_1, \dots, \alpha_n \in E_n} [f(\alpha_1, \dots, \alpha_n) \bullet x_1^{\alpha_1} \bullet \cdots \bullet x_n^{\alpha_n}], \tag{8}$$

where $\bigcirc$ *and* x^α *have the same meanings as in the Theorem.*

P r o o f It suffices to show that the identity operation $y^* = y$ and the above defined partial operations $\circ$ and $\bullet$ fulfil (1). But (5) implies $\omega \bullet \omega = \omega$ and $\omega \circ \varepsilon = \omega$, that is

$$\omega \bullet \beta = \omega \text{ ıor } \beta \in \{\omega, \varepsilon\}, \tag{9}$$

while $\beta \bullet \omega = \omega$ again by (5). Therefore

$$y = \beta_1 \bullet \cdots \bullet \beta_{k-1} \bullet \omega \bullet \beta_{k+1} \bullet \cdots \bullet \beta_n =$$
$$= (\cdots (((y \bullet \beta_1 \bullet \cdots \bullet \beta_{k-1}) \bullet \omega) \bullet \beta_{k+1}) \bullet \cdots) \bullet \beta_n =$$
$$= (\cdots (\omega \bullet \beta_{k+1}) \bullet \cdots) \bullet \beta_n = (\cdots (\omega \bullet \beta_{k+2}) \bullet \cdots) \bullet_n =$$
$$= \cdots = \omega \bullet \beta_n = \omega,$$

whenever $\beta_1, \dots, \beta_{k-1}, \beta_{k+1}, \dots, \beta_n \in \{\omega, \varepsilon\}$. As

$$y \circ \omega \circ \cdots \circ \omega = \omega \text{ and } y \bullet \varepsilon \bullet \cdots \bullet \varepsilon = y$$

follow immediately from (4) and (5), respectively, the left side of (1) becomes

$$\omega \circ \cdots \circ \omega \circ y_i \circ \omega \circ \cdots \circ \omega =$$
$$= (\cdots (((\omega \circ \omega \circ \cdots \circ \omega) \circ y_i) \circ \omega) \circ \cdots) \circ \omega =$$
$$= (\cdots ((\omega \circ y_i) \circ \omega) \circ \cdots) \circ \omega = (\cdots (y_i \circ \omega) \circ \cdots) \circ \omega =$$
$$= y_i \circ \omega \circ \cdots \circ \omega = y_i$$

if $1 < i < p$ and the proof is even simpler for $i=1$ or $i=p$.

Scholium. *Formula* (8) *can be also written in the form given by Carvallo*:

$$(10)\quad f(x_1, \ldots, x_n) = \underset{\alpha_1\in E_1, \ldots, \alpha_n\in E_n}{\bigcirc} [f\,\alpha_1, \ldots, \alpha_n) \cdot \min(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})],$$

where min *refers to the ordering*

$$\leqslant = \{(\omega, \omega), (\omega, \varepsilon), (\varepsilon\ \varepsilon)\}.$$

Proof. We have to establish the identity

$$(11)\qquad y \cdot \beta_1 \cdot \cdots \cdot \beta_n = y \cdot \min(\beta_1, \ldots, \beta_n)$$

for every $y \in E$ and every $\beta_1, \ldots, \beta_n \in \{\omega, \varepsilon\}$. As

$$y \cdot \varepsilon \cdot \cdots \cdot \varepsilon = y = y \cdot \varepsilon = y \cdot \min(\varepsilon, \ldots, \varepsilon),$$

it suffices to prove (11) in the case when $\beta_k = \omega$ for some k. But in this case $y \cdot \min(\beta_1, \ldots, \beta_n) = y \cdot \omega = \omega$, while $y \cdot \beta_1 \cdot \cdots \cdot \beta_n = \omega$ was established in the proof of the above Proposition.

Comments. 1. The remark that $\circ$, $\cdot$ may be partially defined is due to Carvallo [1]; yet he makes the unnecessary assumptions that E is a totally ordered set with ω and ε as least and greatest elements, respectively, and $E_1 = \cdots = E_n = E$.

2. The interpolation formulas III—VIII given by Moisil cannot be obtained from the above Proposition, because if we start with the binary operations $\top$ and $\perp$ and apply the recursion of type (6) we do not obtain the q-ary functions $\top$ and $\perp$. Thus, e. g., $(x_1 \top x_2) \top x_3 = (x_1 \vee x_2)' \vee x'_3$ whereas $(x_1 \top x_2) \top x_3 = (x_1 \vee x_2) \wedge x'_3$; etc.

3. On the other hand, the hypotheses of the Proposition are more simple and natural than those of the Theorem.

4. It is clear from the above discussion and the examples sketched in [1] that our theorem embraces many interpolation formulas widely used in the field of discrete mathematics. However there are also interpolation formulas which do not fall under the scheme suggested by our Theorem, because those formulas are based on quite different ideas; cf. M. Davio, J. — P. Deschamps and A. Thayse [2].

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