PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 25 (39), 1979, pp. 37–44

# ON A CLASS OF BANACH SPACES OF ANALYTIC FUNCTIONS

*Dušan Georgijević*

(Received September 13. 1978)

P r e l i m i n a r i e s In this paper we use the following notation and terminology:

$C^+$ — the upper open half-plane of the complex plane;

$C^-$ — the lower open half-plane;

$R$ — the real line;

$p$ — a finite real number larger than 1; if $1<p<\infty$, then $q$ will denote the number for which $1/p+1/q=1$;

$L^p$ — the space of all measurable functions which are $p$-power integrable on $R$;

$H^p$ — the Hardy space of index $p$ over $C^+$ (consisting of functions $f$ analytic in $C^+$, such that $f(x+iy)\in L^p$ for each $y>0$ and $\|f\|_p=\sup_{y>0}\|f(x+iy)\|_{L^p}<\infty$) ([1], p. 187.); we will identify every $H^p$ function with its boundary function and also the space $H^p$ with the space $\tilde{H}^p$ of boundary functions of $H^p$ functions (as in [1]).

If $E$ is an operator which maps $L^p$ in $L^p$, we will denote with $E^{-1}$ the inverse operator (if it exists) and with $E^*$ the conjugate operator for $E\,(E^*: L^q\to L^q)$.

If $f\in L^p$, $g\in L^q$, then we write

$$\langle f, g\rangle=\int_{-\infty}^{+\infty} f(x)\,\overline{g(x)}\,dx$$

and call this integral inner product of the functions $f$ and $g$ (like in the case $p=2$). If $\langle f, g\rangle=0$, then we will say that the functions $f$ and $g$ are orthogonal: $f\perp g$. If $f\in L^p$, $M\subset L^q$ and, for every $g\in M$ is $f\perp g$, then we say that the function $f$ is orthogonal to the set $M: f \perp M$. If $M \subset L^p$, $N \subset L^q$ and if $f \perp g$ for each $f \in M$ and each $g \perp N$, then we say that the sets $M$ and $N$ are orthogonal: $M \perp N$.

The function

$$k(w, z) = \frac{1}{2\pi i(\bar{w} - z)} \qquad (w, z \in C^+)$$

which has the properties

$$k(w, z) \in H^p, \; w \text{ fixed},$$

and

$$\langle f, k(w, z)\rangle = f(w), \; f \in H^p, \; w \in C^+,$$

([1], p. 195., Th. 11.8.) — will be called reproducing kernel for the space $H^p$.

For $f \in L^p$ we will write $Sf = \bar{f}$. $S$ is a bounded antilinear operator in each $L^p$, with properties:

$S^2 = I$ ($I$ – the identity mapping),

$$\langle Sf, Sg\rangle = \overline{\langle f, g\rangle} \qquad (f \in L^p, \; g \in L^q),$$

$H^p \oplus SH^p = L^p$ (i.e. $\tilde{H}^p \oplus S\tilde{H}^p = L^p$) ([2], p. 151. Th. of M. Riesz). Here $H^p \oplus SH^p$ means that every function $f \in L^p$ can be represented uniquely in the form $f = f_1 + f_2$, $f_1 \in H^p$, $f_2 \in SH^p$. It is also $H^p \perp SH^q$ and $SH^p \perp H^q$. Note also that every $L^p$ function which is orthogonal to $SH^q$ must belong to $H^p$, or, equivalently, that every functino $f$ in $SH^p$ which is orthogonal to $SH^q$ must vanish identically. This follows from the fact that $S_z k(w, z) \in SH^q$ for $w \in C^+$, which gives

$$0 = \langle f, S_z k(w, z)\rangle = \overline{\langle Sf, k(w, z)\rangle} = \overline{Sf(w)}; \; w \in C^+,$$

i.e. $f = 0$.

A function $e$ analytic in $C^+$ is called inner function (for $C^+$) if $|e(z)| \leqslant 1$, $z \in C^+$, and $|e(x)| = 1$ for almost every $x \in R$, where $e(x)$ is the boundary function of $e$ (in $C^+$), i.e. $e(x) = \lim_{y \to 0+} e(x + iy)$ for $a$, $e$. $x \in R$. It follows from the maximum principle that if $e \neq \text{const.}$ then $|e(z)| < 1$ for $z \in C^+$.

Definition and properties of spaces $H^p(E)$. The purpose of this paper is to introduce the spaces $H^p(E)$ and to establish a few of their basic properties.

Definition 1. Let $E$ be a one-to-one bounded linear operator which maps $L^p$ onto $L^p$, and let $E$ have the following properties:

(1) ...... $EH^p \subset H^p$ and $SES = E^{-1}$.

We define $H^p(E)$ as the class of all functions $f$ in $H^p$ for which $ESf \in H^p$.

Note 1. The function $ESf$ belongs to the class $H^p(E)$ whenever $f$ belongs to the class, because of (1).

Note 2. If an operator $E$ is such that the class $H^p(E)$ exists, then the operator $E^{*-1}$ satisfies the conditions for construction of the class $H^q(E^{*-1})$: for if $g \in H^q$, then $E^{*-1}g \in L^q$ and for every $f \in SH^p$ it holds

$$\langle f, E^{*-1}g\rangle = \langle E^{-1}f, g\rangle = \langle SESf, g\rangle = 0,$$

i.e. $E^{*-1}g \in H^q$.

Theorem 1. *$H^p(E)$ is a closed subspace of $H^p$. The function*

(2) $$\ldots K(w, z) \overset{\text{def}}{=} k(w, z) - E_z^{*-1}\overline{E_w k(z, w)} \qquad (w, z \in C^+)$$

*(where $z(w)$ in the index denotes that the operator is applied to a function of the variable $z(w)$) belongs to $H^q(E^{*-1})$ for any fixed $w \in C^+$ and satisfies*

(3) $$\ldots f(w) = \langle f, K(w, z)\rangle, \ f \in H^p(E)$$

*("the relation of reproduction").*

Proof. Let $\{f_n\}_1^\infty$ be a sequence of functions in $H^p(E)$, which converges, in the metric of $H^p$, to a function $f \in H^p$. Since $ESf_n \in H^p$ for $n = 1, 2, \ldots$ and since the operators $S$ and $E$ are bounded, the limit $\lim\limits_{n\to\infty} ESf_n$ exists and lies also in $H^p$. Therefore, as $\lim\limits_{n\to\infty} ESf_n = ESf$, it follows that $ESf \in H^p$. This means that $f \in H^p(E)$, i.e. $H^p(E)$ is a closed subspace of $H^p$.

In order to prove the second part of the statement note that, for any fixed $w \in C^+$, the functional

$$J_w : f \to Ef(w), \quad f \in H^p,$$

is bounded. Indeed, by Hölder's inequality,

$$|Ef(w)| = |\langle Ef, k(w, z)\rangle| \leqslant \|E\| \cdot \|k(w, z)\|_q \cdot \|f\|_p .$$

By the Hahn-Banach theorem, the functional can be extended to the full $L^p$ (without increasing the norm). Then there exists at least a function $G_w \in L^q$ such that

$$Ef(w) = \langle f, G_w\rangle, \quad f \in H^p.$$

Let us determine the projection $g_w$ of the function $G_w$ to $H^q$, in the sense of theorem of M. Riesz ([2], p. 151.). Since

$$G_w - g_w \perp H^p,$$

it holds

(4) $$\ldots Ef(w) = \langle f, g_w\rangle, \quad f \in H^p.$$

Let, for a moment, $f(z) = k(w_1, z)$ ($w_1$ fixed). Substituting this in (4), we obtain

$$E_w k(w_1, w) = \langle k(w_1 z), g_w\rangle = \overline{g_w(w_1)}.$$

Accordingly, $g_w(z)=\overline{E_w k(z, w)}$. Since $E^{*-1} H^q \subset H^q$, it follows

$$E_z^{*-1} \overline{E_w k(z, w)} \in H^q,$$

which gives

$$K(w, z) \in H^q, \quad w \in C^+.$$

But the function $(E^{*-1} S)_z K(w, z)$ also belongs to $H^q$, besause it belongs to $L^q$ and is orthogonal to $SH^p$, as the following calculation shows:

$$f \in SH^p \Rightarrow \langle (E^{*-1} S)_z K(w, z), f\rangle = \langle (E^{*-1} S)_z k(w, z), f)\rangle -$$
$$- \langle S_z \overline{E_w k(z, w)}, f\rangle = \langle ESf, k(w, z)\rangle - \langle Sf, \overline{E_w k(z, w)}\rangle =$$
$$= ESf(w) - E(Sf)(w) = 0,$$

because of (4). So, the function (2) really lies in $H^q(E^{*-1})\ (w \in C^+)$.

It remains to prove „the relation of reproduction" (3). Let $f$ be an arbitrary function in $H^p(E)$. Then we have, for $w \in C^+$,

$$\langle f, K(w, z)\rangle = \langle f, k(w, z)\rangle - \overline{\langle ESf, S_z \overline{E_w k(z, w)}\rangle} = f(w).$$

This completes the proof.

Definition 2. The function (2) will be called reproducing kernel for the space $H^p(E)$.

Note 3. Every function $f$ in $H^p$ can be represented in the form $f=f_1+f_2$, where $f_1 \in H^p$, $f_1 \perp H^q(E^{*-1})$ and $f_2 \in H^p(E)$. This follows from the fact that $E^{-1}f=g_1+g_2$, $g_1 \in H^p$, $g_2 \in SH^p$, by setting $f_1=Eg_1$ and $f_2=Eg_2$. Indeed, $Eg_1 \in H^p$ and if $g \in H^q(E^{*-1})$, then

$$\langle Eg_1, g\rangle = \langle g_1, E^* g\rangle = \langle g_1, SE^{*-1} Sg\rangle = 0,$$

which implies that $f_1 \in H^p$ and $f_1 \perp H^q(E^{*-1})$; further $Eg_2=f-Eg_1 \in H^p$ and $ESEg_2=Sg_2 \in H^p$, which means that $f_2 \in H^p(E)$.

Note 4. Every function $g \in H^q(E^{*-1})$ is the limit, in the metric of $H^q$, of a sequence of linear combinations of functions $K(w, z)$ $w \in C^+$. In other words, the closed span $M$ of the functions $K(w, z)$ in $H^q(E^{*-1})$ is the full space $H^q(E^{*-1})$. This will follow from the Hahn-Banach theorem, if we show that every linear bounded functional on $H^q(E^{*-1})$ which vanishes on $M$ must vanish everywhere on $H^q(E^{*-1})$. Let $F$ be such a functional. We can extend $F$ to the full $L^q$ and then to find a function $f \in L^p$ such that

$F(g)=\langle g, f\rangle$, $g \in L^q$. By Note 3., $f=f_1+f_2+f_3$, where $f_1 \in H^p$, $f_1 \perp H^q(E^{*-1})$, $f_2 \in H^p(E)$ and $f_3 \in SH^p$. If $F(g)=0$ for $g \in M$. then $f \perp M$. But, as $SH^p \perp H^q(E^{*-1})$, this implies that $f_2 \perp M$ and $f_2(w)=\langle f_2, K(w, z)\rangle = 0$, $w \in C^+$, i.e. $f_2=0$. Thus $f$ is orthogonal to $H^q(E^{*-1})$, i.e. $F(g)=0$ for $g \in H^q(E^{*-1})$.

If $H^p(E_1)$ and $H^p(E_2)$ are given spaces and the operator $E_1$ commutes with $E_2$, then the space $H^p(E_1 E_2)$ also exists and it holds

$$H^p(E_j) \subset H^p(E_1 E_2), \quad j=1, 2.$$

Accordingly, to each space $H^p(E)$ one can correspond an increasing sequence of spaces: $\{H^p(E^n)\}_1^\infty$. In the next theorem we give a sufficient condition that the union of these spaces be dense in $H^p$.

Theorem 2. *If the following condition is satisfied*

$$(5) \qquad \ldots \lim_{n\to\infty} \overline{E_z^n E_w^{*-n} k(z, w)} = 0, \; w \in C^+,$$

*in the sense of the weak convergence of a sequence of functions (of z) in $H^p$, then it is*

$$\overline{\bigcup_{n=1}^{\infty} H^p(E^n)} = H^p.$$

Proof. A simple application of the Hahn-Banach theorem (as in Note 4.) shows that it suffices to prove that every $L^q$ function which is orthogonal to $\bigcup_{n=1}^{\infty} H^p(E^n)$ must belong to $SH^q$, or, equivalently, that every $H^q$ function which is orthogonal to $\bigcup_{n=1}^{\infty} H^p(E^n)$ must vanish identically. Let $f \in H^q$ and $f \perp H^p(E^n)$, $n=1, 2, \ldots$. Denote by $K_n(w, z)$ the reproducing kernel for the space $H^q(E^{*-n})$. Then $K_n(w, z) \in H^p(E^n)$, $w \in C^+$. Since for each fixed $w \in C^+$ it holds

$$f(w) = \langle f, k(w, z)\rangle = \langle f, K_n(w, z)\rangle +$$

$$+ \langle f, \overline{E_z^n E_w^{*-n} k(z, w)}\rangle = \langle f, \overline{E_z^n E_w^{*-n} k(z, w)}\rangle \underset{n\to\infty}{\to} 0,$$

the theorem follows.

The main examples of the spaces $H^p(E)$ are those for which the operator $E$ is multiplication by (the boundary function of) an inner function $e$. Such an operator satisfies the conditions for construction of the space $H^p(E)$, for all values of $p$. In this case we will write $H^p(e)$ in place of $H^p(E)$. Moreover, $E^*$ is then multiplication (in $L^q$) by the function $\overline{e(x)} (= e^{-1}(x))$, so that the function

$$K(w, z) = [1 - e(z)\overline{e(w)}] \cdot k(w, z) \qquad (w,z \in C^+)$$

belongs to every $H^p(e)$ and in each of these spaces has the property of reproduction (the relation (3)). The function $K(w, z)$ is positive definite, since it is the reproducing kernel of the Hilbert space $H^2(e)$ ([3]).

In this case is also satisfied the condition (5) in Theorem 2., if $e \neq$const. The condition holds then in the sense of convergence in the metric of $H^p$:

$$(6) \qquad \ldots \|e^n(z)\overline{e^n(w)}\, k(w,z)\|_p = |e(w)|^n \cdot \|k(w, z)\|_p \underset{n\to\infty}{\to} 0, \qquad w \in C^+.$$

The inner function $e$ can be extended outside of $C^+ \cup R$, by setting $e(z) = \overline{e(\bar{z})}^{-1}$ for every $z \in C^-$ such that $e(\bar{z}) \neq 0$. So extended, $e$ is analytic at those points $z \in C^-$ for which $e(\bar{z}) \neq 0$ and has the same boundary function $e(x)$ in $C^-$ as in $C^+$:

$$e(x) = \lim_{y \to 0_-} e(x+iy) \quad \text{a.e. on } R.$$

The function $e$ can be analytic also at some points on $R$. We will denote by $A$ the set of all those points of the complex plane at which $e$ is analytic.

Theorem 3. *If $E$ is multiplication in $L^p$ by an inner function $e$, then each function $f \in H^p(e)$ can be extended to a function analytic in $A$: Convergence in the metric of $H^p(e)$ implies the uniform convergence on compact subsets of $A$. Each functional*

$$J_w : f \to f(w), \quad f \in H^p(e), \quad w \in A,$$

*is then bounded.*

Proof. We will extend any function $f \in H^p(e)$ by setting

$$f(z) \overset{\text{def.}}{=} e(z)\, \overline{ESf)(\bar{z})}, \quad \text{for } z \in C^- \wedge e(\bar{z}) \neq 0.$$

Since the $H^p$ function $ESf$ is analytic in $C^+$, the (extended) function $f$ is analytic in $\{z/z \in C^- \wedge e(\bar{z}) \neq 0\}$. It has the same boundary function in $C^-$ as in $C^+$.

The function

$$K(w, z) = [1 - e(z)\, \overline{e(w)}] \cdot k(w, z)$$

is analytic as a function of $z$ in the whole set $A$, if $w$ is a fixed point in $A$. We will show that $K(w, z) \in H^p(e)$, $w \in A$. Let $w \in A \cap C^-$. Since

$$K(w, z) = \overline{e(w)}\, (ES)_z\, K(\bar{w}, z),$$

it follows that $K(w, z) \in H^p(e)$ (for every $p$). The relation of reproduction also holds:

$$\langle f, K(w, z)\rangle = e(w) \overline{\langle ESf, K(\bar{w}, z)\rangle} = e(w)\, \overline{(ESf)(\bar{w})} = f(w).$$

This implies also that the functional $J_w$ is bounded.

Let now $w \in A \cap R$. We will show that $K(w, z) \in H^p$ and $(ES)_z K(w, z) \in H^p$. The point w can be surrounded by a closed square of side $2r$ in which $K(w, z)$ is analytic as a function of $z$, therefore also bounded, by a constant $M$. This implies that for $0 \leqslant y \leqslant r$

$$\int_{-\infty}^{+\infty} |K(w, t+iy)|^p\, dt = \int_{w-r}^{w+r} + \int_{|w-t|>r} \leqslant$$

$$\leqslant 2M^p r + \left(\frac{1 + |e(w)|}{2\pi}\right)^p \int_{|w-t|>r} \frac{dt}{|w-t|^p},$$

and for $y>r$

$$\int_{-\infty}^{+\infty} |K(w, t+iy)|^p \leqslant \left(\frac{1+|e(w)|}{2\pi}\right)^p \int_{-\infty}^{+\infty} \frac{dt}{|w-t-ri|^p}.$$

Thus $K(w, z) \in H^p$, $w \in A \cap R$. The proof of $(ES)_z K(w, z) \in H^p$ is quite similar, because of

$$(ES)_z K(w, z) = \frac{e(z)-e(w)}{2\pi i(z-w)}.$$

Now we extend each function $f \in H^p(e)$ to $A \cap R$ by setting

$$f(w) \overset{\text{def.}}{=} \langle f, K(w, z) \rangle, \quad w \in A \cap R.$$

Our next intention is to show that the convergence in $H^p(e)$ implies the uniform convergence on compact subsets of $A$. Let $B$ be an arbitrary compact subset of $A$. Surround $B$ by an open subset $B_1$ of $A$ such that its closure $\overline{B_1}$ is a compact subset of $A$. (This is possible because $B$ is a compact and $A$ is an open set.) The function $K(w, z)$ is analytic in $A$ as a function of $z$ $(w \in A)$ and in $SA$ as a function of $\bar{w}$ $(z \in A)$. (Here is $SA = \{z/\bar{z} \in A\}$.) Hence it is analytic at every point $(\bar{w}, z) \in SA \times A$ as a function of two variables ([4], Th. of Hartogs, p. 284.). Such a function must be continuous on the compact $SB \times \overline{B_1}$, and therefore bounded:

$$|K(w, z)| \leqslant M, \quad w \in B, \quad z \in \overline{B_1}.$$

The sets $CB_1$ and $B$ are closed and disjoint and $B$ is compact, so the distance $d$ between $CB_1$ and $B$ is positive. The norm of $K(w, z)$ in $L^p$ is bounded for $w \in B$, by a constant $N_p$ independently on $w$:

$$\|K(w, z)\|_p^p = \int_{R \cap \overline{B_1}} |K(w, t)|^p\, dt + \int_{R \setminus \overline{B_1}} |K(w, t)|^p\, dt \leqslant$$

$$\leqslant M^p \int_{R \cap \overline{B_1}} dt + \left(\frac{1+\max_{w \in B} |e(w)|}{2\pi}\right)^p \left[\frac{1}{d^p} \int_{[-2m,\, 2m] \setminus \overline{B_1}} dt + \right.$$

$$\left. + \int_{(R \setminus [-2m,\, 2m]) \setminus \overline{B_1}} \frac{dt}{(|t|-m)^p}\right] = N_p^p, \quad \text{where } m = \max_{w \in B} |w|.$$

Let a sequence $\{f_n\}_1^\infty$ in $H^p(e)$ converge to $f$ and let again $B$ be a compact subset of $A$. Then

$$|f(w) - f_n(w)| = |\langle f - f_n, K(w, z) \rangle| \leqslant$$

$$\leqslant \|f - f_n\|_p \cdot \|K(w, z)\|_q \leqslant \|f - f_n\| \cdot N_q, \quad w \in B,$$

which shows that the sequence $\{f_n\}$ converges uniformly on $B$.

It remains only to show that any function $f \in H^p(e)$ is analytic in $A \cap R$. The function $f$ is the limit (in the metric of $H^p(e)$) of a sequence of linear combinations of functions $K(w, z)$, $w \in C^+$ (by Note 4.). By what we just showed, the sequence converges to $f$ uniformly on compact subsets of $A$. Since all members of the sequence are analytic at points $z \in A \cap R$, the function $f$ is analytic at points $z \in A \cap R$.

The theorem is completely proved.

## REFERENCES:

[1] P. L. Duren; *Theory of $H^p$ spaces*, Academic Precc, New York and London, 1970.

[2] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[3] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337—404.

[4] Б. В. Шабат, *Введение в комплексный анализ*, Наука, Москва, 1969.