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ON MULTIPLICATION FORMULAE FOR FOX'S *H*-FUNCTION

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Abstract **In the present paper, we have derived two multiplication formulae for Fox's H-function using the fractional derivative operators D_x^{α} and $D_{K,\,\alpha\,x}^{n}$. We have also obtained multiplication formulae for Meijer's G-function and the hypergeometric functions as particular cases of one of the formulae.**

1. Introduction

The *H*-function introduced by Fox [2] is defined and represented in the following manner [3]:

$$H_{p,\,q}^{m,\,n}\left[z\,\middle|\,\begin{matrix}(a_j,\ \alpha_j)_{1,\,p}\\(b_j,\ \beta_j)_{1,\,q}\end{matrix}\right]$$

(1.1)

$$=\frac{1}{2\pi i}\int_c \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\,z^s\,ds$$

where $z\neq 0$, m, n, p, q, are integers satisfying $1\leqslant m\leqslant q$, $0\leqslant n\leqslant p$, α_j $(j=1, 2, \ldots, p)$, $\beta_j\,(j=1, 2, \ldots, q)$ are positive numbers and $a_j\,(j=1, \ldots, p)$, $b_j\,(j=1, 2, \ldots, q)$ are complex numbers. The contour c is a straight line running from $\sigma-i\infty$ to $\sigma+i\infty$ in such a manner that the poles of $\Gamma(b_j-\beta_j s)$ $j=1, 2, \ldots m$ lie to the right and all the poles of $\Gamma(1-a_j+\alpha_j s)\;j=1, 2, \ldots, n$ lie to the left of the contour. All the poles are assumed to be simple.

The conditions under which the integral (1.1) converges asymptotic expansion of the *H*-function and its particular cases can be referred to in a paper by Gupta and Jain [3].

2. In a recent paper Misra [4] has defined the fractional derivative operators in the following manner:

$$D_x^{\alpha}(x^{\mu-1})=\frac{d^{\alpha}}{dx^{\alpha}}x^{\mu-1}=\frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)}x^{\mu-\alpha-1},\quad \alpha\neq\mu \tag{2.1}$$

$$D_{K,\,\alpha,\,x}(x^{\mu})=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}x^{\mu+K},\quad \alpha\neq\mu+1 \tag{2.2}$$

α and K are not necessarily integers

$$D^{n}_{K,\,\alpha,\,x}(x)=\prod_{r=0}^{n-1}\left[\frac{\Gamma(\mu+1+rK)}{\Gamma(\mu+1+rK-\alpha)}\right]x^{\mu+nK} \tag{2.3}$$

when $K=\alpha$, (2.3) becomes

$$D^{n}_{\alpha,\,\alpha,\,x}(x^{\mu})=\frac{\Gamma[\mu+(n-1)\alpha+1]}{\Gamma(\mu+1-\alpha)}x^{\mu+n\alpha} \tag{2.4}$$

The following two theorems given by Dube [2] are required to establish our main results:

Theorem 1.

$$D^{m}_{K,\,\lambda-\mu,\,t}[t^{\lambda-1}f(xt)]=\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pK)}{\Gamma(\mu+pK)}\right]t^{\lambda-1+mK}\cdot$$
$$\cdot\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,{}_{m+1}F_m(-n,\lambda,\lambda+K,\ldots,\lambda+\overline{m-1}\,K;$$
$$\mu,\mu+K,\ldots,\mu+\overline{m-1}\,K;t)\;D_x^{n}[f(x)] \tag{2.5}$$

Theorem 2.

$$D^{m}_{K,\,\lambda-\mu,\,t}[t^{\lambda}f(xt)]=\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pK)}{\Gamma(\lambda+pK)}\right]t^{\lambda-1+mK}\cdot$$
$$\cdot\sum_{n=0}^{\infty}\frac{(-t)^{-n}}{n!}\prod_{p=0}^{m-1}\left\{\frac{(1-\mu-pK)_n}{(1-\lambda-pK)_n}\right\}\cdot$$
$$\cdot\,{}_{m+1}F_m(-n,\lambda-n,\lambda-n+K,\ldots,\lambda-n+\overline{m-1}\,K;$$
$$\mu-n,\mu-n+K,\ldots,\mu-n+\overline{m-1}\,K;t)\;D_x^{n}[x^{n}f(x)] \tag{2.6}$$

where the fractional operators occurring in (2.5) *and* (2.6) *have been defined by the equations* (2.1) *to* (2.4).

3. Main results

Here we derive the required multiplication formulae for the Fox's H-function, first by using theorem 1 and then theorem 2. In Theorem 1, let us take

$$f(x)=x^{-bQ}H_{P,Q}^{M,N}\left[x^{\beta Q}\left|\begin{matrix}(a_j,\ \alpha_j)_1,\ P\\(b_j,\ \beta_j)_1,\ Q\end{matrix}\right.\right]$$

so that the right hand side of (2.5) becomes

$$\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pK)}{\Gamma(\mu+pK)}\right]\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,{}_{m+1}F_m(-n,\ \lambda,\ \lambda+K,\ \ldots,$$

$$\lambda+\overline{m-1}\,K;\ \mu,\ \mu+K,\ \ldots,\ \mu+\overline{m-1}\,K:\ t)\,t^{\lambda-1+mK}.$$

$$\cdot D_x^n\left[x^{-bQ}H_{P,Q}^{M,N}\left[x^{\beta Q}\left|\begin{matrix}(a_j,\ \alpha_j)_1,\ P)\\(b_j,\ \beta_j)_1,\ Q)\end{matrix}\right.\right]\right]$$

Now, with the help of (1.1) and (2.1), the above expression reduces to the following value:

$$\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pK)}{\Gamma(\mu+pK)}\right]\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,{}_{m+1}F_m(-n,\ \lambda,\ \lambda+k,\ \ldots,$$

$$\lambda+\overline{m-1}\,K;\ \mu,\ \mu+K,\ \ldots,\ \mu+\overline{m-1}\,K;\ t)\,t^{\lambda-1+mK}x^{-bQ}.$$

$$\cdot H_{P,Q}^{M,N}\left[x^{\beta Q}\left|\begin{matrix}(a_j,\ \alpha_j)_1,\ P\\b_j,\ \beta_j)_{1,\ Q-1},\ (b_Q+n,\ \beta_Q)\end{matrix}\right.\right]\tag{3.1}$$

Also, the left hand side of the theorem 1, becomes

$$D_{K,\ \lambda-\mu,\ t}^{m}\left[t^{\lambda-1}x^{-bQ}t^{-bQ}\,H_{P,Q}^{M,N}\left[(xt)^{\beta Q}\left|\begin{matrix}(a_j,\ \alpha_j)_1,\ P\\(b_j,\ \beta_j)_1,\ Q\end{matrix}\right.\right]\right]$$

In the above expression, we use (1.1) and (2.3), we get the following value of it

$$x^{-bQ}t^{\lambda-bQ-1+mK}H_{P+m,\ Q+m}^{M,\ N+m}\left[(xt)^{\beta Q}\left|\begin{matrix}(1-\lambda-pK+b_Q,\ \beta_Q)_{p=0,\ m-1},(a_j,\ \alpha_j)_1,\ P\\(b_j\ \beta_j)_{1,\ Q},\ (1-\mu-pK+b_Q,\ \beta_Q)_{p=0,\ m-1}\end{matrix}\right.\right]\tag{3.2}$$

Hence, from (3.1) and (3.2) we get the following multiplication formula:

$$H_{P+m,\, Q+m}^{M,\, N+m}\left[(xt)^{\beta_Q}\left|\begin{matrix}(1-\lambda-pK+b_Q,\ \beta_Q)_{p=0,\ m-1},(a_j,\ \alpha_j)_{1}\ P\\ (b_j,\ \beta_j)_{1,\ Q},\ (1-\mu-pK+b_Q,\ \beta_Q)_{p=0,\ m-1}\end{matrix}\right.\right]$$

$$=\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pK)}{\Gamma(\mu+pK)}\right]t^{b_Q}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\,{}_{m+1}F_m(-n,\ \lambda,\ \lambda+K,\ \ldots,$$

$$\lambda+\overline{m-1}\,K;\ \mu,\ \mu+K,\ \ldots,\ \mu+\overline{m-1}\,K;\ t)\cdot$$

$$\cdot H_{P,\,Q}^{M,\,N}\left[x^{\beta_Q}\left|\begin{matrix}(a_j,\ \alpha_j)_1,\ P\\ (b_j,\ \beta_j)_{1,\ Q-1},\ (b_Q+n,\ b_Q)\end{matrix}\right.\right] \tag{3.3}$$

Next, in Theorem 2, we put

$$f(x)=x^{-a_1}H_{P,\,Q}^{M,\,N}\left[x^{a_1}\left|\begin{matrix}(a_j,\ \alpha_j)_1,\ P\\ (b_j,\ \beta_j)_1,\ Q\end{matrix}\right.\right],$$

and proceed in a manner similar to (3.3), we get, after a little simplification, the following multiplication formula for Fox's H-function:

$$H_{P+M,\, Q+m}^{M,\, N+m}\left[(xt)^{\alpha_1}\left|\begin{matrix}(a_1-\lambda-pK,\ \alpha_1)_{p=0,\ m-1},\ (a_j,\ \alpha_j)_1,\ P\\ (b_j,\ \beta_j)_{1,\ Q},\ (a_1-\mu-pK,\ \alpha_1)_{p=0,\ m-1}\end{matrix}\right.\right]$$

$$=\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pK)}{\Gamma(\mu+pK)}\right]t^{a_1-1}\sum_{n=0}^{\infty}\frac{(-t)^{-n}}{n!}\prod_{p=0}^{m-1}\left\{\frac{(1-\mu-pK)_n}{(1-\lambda-pK)_n}\right\}\cdot$$

$$\cdot\,{}_{m+1}F_m(-n,\ \lambda-n,\ \lambda-n+K,\ \ldots,\ \lambda-n+\overline{m-1}\,K;\ \mu-n,$$

$$\mu-n+K,\ \ldots,\ \mu-n+\overline{m-1}\,K;\ t)\ H_{P,\,Q}^{m,\,N}\left[x^{\alpha_1}\left|\begin{matrix}(a_1-n,\ \alpha_1),\ (a_j,\ \alpha_j)_2,\ P\\ (b_j,\ \beta_j)_{1,\ Q}\end{matrix}\right.\right] \tag{3.4}$$

4. Particular cases

If in the multiplication formula (3.3) we put all $\alpha_j=\beta_j=1$, we get, the following multiplication formula involving Meijer's G-function:

(i)

$$G_{P+m,\, Q+m}^{M,\, N+m}\left[xt\left|\begin{matrix}(1-\lambda-pK+b_Q)_{p=0,\ m-1},\ (a_j)_1,\ P\\ (b_j)_{1,\ Q},\ (1-\mu-pK+b_Q)_{p=0,\ m-1}\end{matrix}\right.\right]$$

$$=\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pK)}{\Gamma(\mu+pK)}\right]t^{b_Q}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\,{}_{m+1}F_m(-n,\ \lambda,\ \lambda+K),$$

$$\ldots,\ \lambda+\overline{m-1}\,K;\ \mu,\ \mu+K,\ \ldots,\ \mu+\overline{m-1}\,K;\ t)\cdot$$

$$G_{P,\,Q}^{M,\,N}\left[x\left|\begin{matrix}(a_j)_1,\ P\\ (b_j)_{1,\ Q-1},\ b_{Q+n}\end{matrix}\right.\right] \tag{4.1}$$

(ii) If we put all $\alpha_j = \beta_j = 1$, replace Q by $Q+1$, $b_1 = 0$, and use a known result [3, 600 (4.6)], we get, after a little simplification, the following resu'ı involving hypergeometric function:

$$
\begin{aligned}
&{}_{P+m}F_{Q+m}[(\lambda+pK+b_1-1)_{p=0,\ m-1},\ (a_j)_1,\ P;\ (b_j)_{1,Q},\\
&(\mu+pK+b_1-1)_{p=0,\ m-1};\ xt]\\
&=\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pK)}{\Gamma(\mu+pK)}\,\frac{\Gamma(\mu+pK+b_1-1)}{\Gamma(\lambda+pK+b_1-1)}\right]t^{1-b_1}\,.\\
&\cdot\sum_{n=0}^{\infty}\frac{(-t)^n}{n!\,\Gamma(b_1-n)}\,{}_{m+1}F_m(-n,\ \lambda,\ \lambda+K,\ \ldots,\ \lambda+\overline{m-1}\,K;\ \mu,\\
&\mu+K,\ \ldots,\ \mu+\overline{m-1}\,K;\ t)\ {}_PF_Q[(a_j)_{1,\ P};\ (b_1-n,\ (b_j)_{2,\ Q};\ x]
\end{aligned}
\tag{4.2}
$$

(iii) If in (4.2), we put $P=Q=m=1$, we get, after a little simplification, the following interesting formula:

$$
\begin{aligned}
&{}_2F_2[\lambda+b-1,\ a;\ b,\ \mu+b-1;\ xt]\\
&=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\,\frac{\Gamma(\mu+b-1)}{\Gamma(\lambda+b-1)}\,t^{1-b}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!\,\Gamma(b-n)}\times\\
&\times{}_2F_1(-n,\ \lambda;\ \mu;\ t)\ {}_1F_1(a,\ b-n;\ x)
\end{aligned}
\tag{4.3}
$$

Particular cases for other simpler functions can also be obtained from (3.3) on account of the most general nature of Fox's H-function, but we do not mention them here for want of space.

Also, particular cases for multiplication formula (3.4) can be obtained in a manner similar to that of (3.3).

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