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ON ANTI-INVERSE SEMIGROUPS

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In [1], anti-inverse semigroups are studied. Here, for the class of anti--inverse semigroups, we determine basic class in the sense of E. S. Ljapin (Chapter VIII § 5 [7]). Then, using the theory of R. Croisot (Chapter 4 § 4.1. [3] or [4]) and the relations of J. Green [5], we prove various theorems of decompositions of anti-inverse semigroups.

A semigroup S is regular (left, right regular) if for every $x \in S$ there exists a S so that $x = xax\,(x = ax^2,\ x = x^2 a)$ is satisfied (pp. 163. [3]).

1. For elements x and y of a semigroup S, we say that they are mutually anti-inverse if the following conditions hold

$$xyx = y \quad \text{and} \quad yxy = x.$$

A semigroup S is anti-inverse if for every $x \in S$ there exists its anti-inverse element $y \in S$. By $\mathcal{A}$ we denote the class of anti-inverse semigroups.

In [1], the following theorem, which we shall use many times in the paper, is proved.

Theorem 1.1. *Let S be a semigroup. Then*

$$S \in \mathcal{A} \Leftrightarrow (\forall x \in S)\,(\exists y \in S)\,(x^2 = y^2,\ yx = x^3 y,\ x^5 = x).$$

2. Immediately, on the basis of the definition and the Theorem 1.1, follows the

Theorem 2.1. *Let S be a semigroup. Then, the following conditions are equivalent:*

(i) $S \in \mathcal{A}$.

(ii) $(\forall x \in S)\,(\exists y \in S)\,(yxy = x,\ x^2 = y^2)$.

(iii) $(\forall x \in S)\,(\exists y \in S)\,(xyx = y,\ x^2 = y^2,\ x^5 = x)$.

(iv) $(\forall x \in S)\,(\exists y \in S)\,(yxy = x,\ y^2 = (xy)^2)$.

Corollary 2.1. *Let S be a cancellative semigroup. Then, the following conditions are equivalent*:

(i) $S \in \mathcal{A}$.

(ii) $(\forall x \in S)\ (\exists y \in S)\ (xyx = y,\ x^2 = y^2)$.

(iii) $(\forall x \in S)\ (\exists y \in S)\ (x^2 = y^2,\ x^2 = (xy)^2)$.

(iv) $(\forall x \in S)\ (\exists y \in S)\ (xy = y^3 x,\ x^2 = y^2)$.

(v) $(\forall x \in S)\ (\exists y \in S)\ (yx = x^3 y,\ x^2 = y^2)$.

(vi) $(\forall x \in S)\ (\exists y \in S)\ (xy = y^3 x,\ xy = yx^3)$.

(vii) $(\forall x \in S)\ (\exists y \in S)\ (xyx = y,\ x^2 = (xy)^2)$.

(viii) $(\forall x \in S)\ (\exists y \in S)\ (xy = y^3 x,\ y^2 = (xy)^2)$.

(ix) $(\forall x \in S)\ (\exists y \in S)\ (yx = x^3 y,\ x^2 = (xy)^2)$.

3. Let P be a nonempty subset of a semigroup S. Denote by $[P]$ the subsemigroup of the semigroup S generated by P.

Theorem 3.1. *Let S be a semigroup. Then*

$$S \in \mathcal{A} \Leftrightarrow (\forall x \in S)\ (\exists y \in S)\ ([\{x, y\}] \in \mathcal{A}).$$

Proof. The proof is similar to the proof of Theorem 3.3. [1].

Corollary 3.1. *For $S = G$, where G is a group, one obtains Theorem* 3.3. [1].

Corollary 3.2. *Let S be a semigroup. Then $S \in \mathcal{A}$ if and only if S is the union of anti-inverse subgroups.*

Proof. Let $S \in \mathcal{A}$. Then, for arbitrary $x \in S$, there exists an anti-inverse element $y \in S$. Hence $[\{x, y\}]$ is anti-inverse subgroup of S (Corollary 3.1. (i) and (ii) [1]) which covers the element x. Thus, S is the union of anti-inverse subgroups.

Conversely, let $S = \bigcup_{i \in I} G_i$, where G_i are anti-inverse subgroups of S. Then, for $x \in S$, there exists $i \in I$ such that $x \in G_i$ and hence there exists an anti-inverse element $y \in G_i$. Therefore, $S \in \mathcal{A}$.

Denote by $\mathcal{B}$ the class consisting of the trivial group, cyclic group of order 2 and of quaternion group.

Theorem 3.2. *Let G be a group. Then, $G = \mathcal{A}$ if and only if G is the union of a subgroup which belongs to the class $\mathcal{B}$.*

Proof. Let $G \in \mathcal{A}$. Then, on the basis of the Corollary 3.1, the group G is covered by groups from the class consisting of the trivial group, cyclic group of order 2, Klein's group and the group of quaternions. However the Klein's groups are covered by cyclic groups of order 2. Therefore, G is the union of subgroup from the class $\mathcal{B}$.

Conversely, let G be the union of subgroups from the class $\mathcal{B}$. Then, any $x \in G$ is in a subgroup of G, which belongs to $\mathcal{B}$, and the groups from the class $\mathcal{B}$ are anti-inverse (Corollaries 3.1. (i) and (ii) [1]). Hence, there exists an anti-inverse element for x. Therefore, $G \in \mathcal{A}$.

This proves the theorem.

The following definition is given by E. S. Ljapin (Chapter VIII § 5 [7]).

Definition 3.1. *Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ be three classes of semigroups such that $\mathcal{M} \subset \mathcal{N} \subset \mathcal{P}$. The class $\mathcal{M}$ is basic class for the class $\mathcal{N}$, relatively to the $\mathcal{P}$, if the following conditions hold:*

a) *Every semigroup from $\mathcal{N}$ can be represented as the union of its subsemigroups which are from the class $\mathcal{M}$.*

b) *Every semigroup from $\mathcal{P}$ which can be represented as the union of its subsemigroups from $\mathcal{M}$ is in $\mathcal{N}$.*

c) *Any subclass $\mathcal{M}'$ of the class $\mathcal{M}$ does not satisfy the condition* a).

For the class of semigroups having the basic class in the sense of the previous definition, we shall say to have basic class in the Ljapin's sense.

If $\mathcal{M} = \mathcal{B}$, $\mathcal{N} = \mathcal{A}$ and if $\mathcal{P}$ is the class of all semigroups, then we have

Theorem 3.3. *The class $\mathcal{B}$ is a basic class in the sense of Ljapin for the class $\mathcal{A}$ relatively to $\mathcal{P}$.*

Proof. The condition a) and b) from the Definition 3.1 are satisfied in virtue of the Corollary 3.2 and the Theorem 3.2. The condition c) is satisfied obviously.

Now we shall give one more characterisation of anti-inverse semigroups.

Theorem 3.4. *Let S be a semigroup. Then, $S \in \mathcal{A}$ if and only if S is the union of disjoint anti-inverse subsemigroups.*

Proof. Let $S \in \mathcal{A}$. Then $S = \bigcup_{i \in I} S$.

Conversely, the proof is similar to that one of the Corollary 3.2.

4. Denote by $L(a)$, $R(a)$, $J(a)$ the left, right and two-sided principal ideal of a semigroup S generated by an element $a \in S$. As the semigroup $S \in \mathcal{A}$ is regular (Corollary 2.1. (i) [1]), we have

$$L(a) = Sa, \quad R(a) = aS, \quad J(a) = SaS \quad \text{(Chapter II [6])}.$$

Lemma 4.1. *Let $y \in S$ be an anti-inverse element for the element x of the regular semigroup S. Then*

$$L(x) = L(y), \quad R(x) = R(y), \quad J(x) = J(y).$$

Proof. Let x and y be mutually anti-inverse elements of a regular semigroup S. Then we have

(4.1) $$L(x)=Sx=S(yxy)=(Syx)y\subset Sy=L(y),$$

(4.2) $$L(y)=Sy=S(xyx)=(Sxy)x\subset Sx=L(x),$$

From (4.1) and (4.2) follow $L(x)=L(y)$.

Similarly, we have $R(x)=R(y)$.

By using these results we have

$$J(x)=SxS=L(x)S=L(y)S=SyS=J(y).$$

Corollary 4.1. *An ideal* (left, right, two-sided) *of the regular semigroup S generated by a set A_x of the anti-inverse elements is the same as the ideal generated by the element x.*

As for each element x from the anti-inverse semigroup S we have $x=x^5$, then $x^2x^3=x^3x^2=x$ and thus the anti-inverse semigroup is left and right regular. From this

(4.3) $$L(x)=L(x^2), \quad \text{(pp. 163—164, [3])}.$$

Lemma 4.2. *Elements x and y of a semigroup $S\in\mathcal{A}$ for which is*

(4.4) $$x^2=y^2$$

generate the same (left, right, two-sided) *principal ideal.*

Proof. Let for the elements x and y of a semigroup $S\in\mathcal{A}$ the condition (4.4) hold. Then $L(x)=Sx=Sx^2=Sy^2=Sy=L(y)$.

Similarly we have $R(x)=R(y)$ and $J(x)=J(y)$.

Corollary 4.2. *Elements of a semigroup S which have the same unity generate a same* (left, right, two-sided) *principal ideal.*

The relations on a semigroup S definied by

(i) $a\mathcal{L}b \Leftrightarrow L(a)=L(b)$,

(ii) $a\mathcal{R}b \Leftrightarrow R(a)=R(b)$,

(iii) $\mathcal{H}=\mathcal{R}\cap\mathcal{L}$

are equivalence relations [5]. Denote by L_a, R_a and H_a $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{H}$-class, respectively, which contain the element $a\in S$, then $H_a=R_a\cap L_a$.

Lemma 4.3. *Let $S\in\mathcal{A}$. Then L_a and R_a are anti-inverse subsemigroups of the semigroup S.*

Proof. Let $S\in\mathcal{A}$. Then S is left regular, so an arbitrary $\mathcal{L}$-class L_a of the semigroup S is subsemigroup of S (Theorem 4.2. [3]). Let $x\in L_a$, then

(4.5) $$L(x)=L(a).$$

For $x\in L_a$ there exists an anti-inverse element $y\in S$, so

(4.6) $$L(x)=L(y) \qquad \text{(Lemma 4.1)}.$$

From (4.5) and (4.6), we have $L(y)=L(a)$, i.e. $y\in L_a$.

Thus, $L_a\in\mathcal{A}$. Similarly, $R_a\in\mathcal{A}$.

Theorem 4.1. *Let S be a semigroup. Then $S\in\mathcal{A}$ if and only if each $\mathcal{L}$-class of S is anti-inverse subsemigroup.*

Proof. Let $S\in\mathcal{A}$. Then each $\mathcal{L}$-class of the semigroup S is anti-inverse subsemigroup (Lemma 4.3.).

Conversely, let each $\mathcal{L}$-class of S is an anti-inverse subsemigroup. Then the semigroup S is the union of anti-inverse subsemigroup. Hence, by Theorem 3.4, we have $S\in\mathcal{A}$.

Corollary 4.3. *Let S be a semigroup. Then, the following conditions are equivalent:*

(i) $S\in\mathcal{A}$.

(ii) *Every $\mathcal{L}$-class of the semigroup S is a left simple anti-inverse subsemigroup.*

(iii) *S is the union of the disjoint left simple anti-inverse subsemigroups.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 4.1 and Theorem 4.2. [3]. (ii) $\Rightarrow$ (iii) follows immediately. (iii) $\Rightarrow$ (i) follows from Theorem 3.4.

Theorem 4.2. *Let S be a semigroup. Then $S\in\mathcal{A}$ if and only if S is left regular and each its left ideal is an anti-inverse subsemigroup.*

Proof. Let $S\in\mathcal{A}$. Then S is left regular and each left ideal is an anti-inverse subsemigroup. (Lemma 1. [2]).

Conversely, let S be a left regular semigroup. Then each $\mathcal{L}$-class L_a of the semigroup S is a subsemigroup of S (Theorem 4.2. [3]). Since $L_a\subset L(a)\in\mathcal{A}$, then for $x\in L_a$ there exists an anti-inverse element $y\in L(a)$. For the elements x and y, we have $x^2=y^2$ (Theorem 1.1), so $L(x)=L(x^2)=L(y^2)=L(y)$, (since S is left regular, pp. 164, [3]). Hence, $y\in L_a$, i.e. every $\mathcal{L}$-class L_a of the semigroup S is an anti-inverse subsemigroup. Hence, by Theorem 4.1 one obtains $S\in\mathcal{A}$.

Theorem 4.3. *Let S be a semigroup. Then $S\in\mathcal{A}$ if and only if S is the union of disjoint anti-inverse subgroups.*

P r o o f. Let $S \in \mathcal{A}$. Then S is left and right regular, which means that S is the union of disjoint groups H_a (Theorem 4.3. [3]). Let us check that H_a is anti-inverse group. Let $x \in H_a$, then $x \in R_a$ and $x \in L_a$. For the element $x \in R_a$ there is its anti-inverse element $y \in R_a$ (Lemma 4.3), so $x^2 = y^2$ (Theorem 1.1). Hence $L(x) = L(y)$ (Lemma 4.2), i.e. $y \in L_a$. So H_a is an anti-inverse group, hence S is the union of disjoint anti-inverse subgroups.

The converse is proved similarly to the converse of the Corollary 3.2.

C o r o l l a r y 4.4. *Let S be a semigroup. Then, the following conditions are equivalent*:

(i) $S \in \mathcal{A}$.

(ii) *S is regular and left regular and each its right ideal is an anti-inverse subsemigroup.*

(iii) *S is regular and right regular and each its left ideal is an anti-inverse subsemigroup.*

P r o o f. From the Theorem 4.3, the Theorem 4.3 [3] and the Theorem 4.1 follow the assertion immediately.

The Theorem 4.3 may be restated as

T h e o r e m 4.4. *Let S be a semigroup. Then* $S \in \mathcal{A}$ *if and only if each* $\mathcal{H}$*-class of the semigroup S is anti-inverse.*

L e m m a 4.4. *Let* $S \in \mathcal{A}$. *Then the relation*

$$x \sim y \Leftrightarrow J(x) = J(y),$$

is a congruence relation on S and the classes are the simple subsemigroups.

P r o o f. Let $S \in \mathcal{A}$, then S is intra-regular (Corollary 2.1. (vii) [1]), so each its two-sided ideal is semiprime (Lemma 4.1. [3]). From this follows that $\sim$ is a congruence relation on S and each class is a simple subsemigroup (Lemma 3. [4]).

P r o p o s i t i o n 4.1. *Let S be a semigroup. Then the following conditions are equivalent*:

(i) $S \in \mathcal{A}$.

(ii) *S is the union of simple anti-inverse subsemigroups.*

(iii) *S is intra-regular and each its two-sided ideal is an anti-inverse subsemigroup.*

P r o o f. (i) $\Leftrightarrow$ (ii). Let $S \in \mathcal{A}$. Then. by Lemma 4.4. $\sim$ is a congruence relation on S and the classes $C_a (a \in S)$ are simple subsemigroups. Let us prove that C_a is anti-inverse. Indeed, let $x \in C_a$, then there exists $y \in S$, so that $xyx = y$ and $yxy = x$, so $J(x) = SxS = SyS = J(y)$ (Lemma 4.1), i.e. $y \in C_n$.

The converse is proved immediately.

(i) $\Leftrightarrow$ (iii). Let $S \in \mathcal{A}$. Then S is intra-regular (Corollary 2.1. (vii) [1]) and every its two-sided ideal is anti-inverse (Lemma 1. [2]).

Conversely, since S is an intra-regular semigroup, then each its two-sided ideal is semiprime (Lemma 4.1. [3]).

Hence $\sim$ is a congruence relation on S and the classes $C_a (a \in S)$ are simple subsemigroups of S (Lemma 3. [4]). As $C_a \subset J(a) \in \mathcal{A}$, then for $x \in C_a$ there is its anti-inverse element $y \in J(a)$. As we have, $J(x) = J(y)$ (Lemma 4.1), then $y \in C_a$. Hence $C_a \in \mathcal{A}$ and C_a is simple, so (iii) $\Rightarrow$ (ii). As we have proved (ii) $\Rightarrow$ (i), then (iii) $\Rightarrow$ (i).

This completes the proof.

Theorem 4.5. *Let S be a semigroup. Then $S \in \mathcal{A}$ if and only if S is the union of completely simple anti-inverse subsemigroups.*

Proof. Let $S \in \mathcal{A}$. Then by Proposition 4.1 the semigroup S is the union of simple anti-inverse subsemigroups. Hence, by Corollary 3.2 and the Theorem 4.5. [3] we have that S is the union of completely simple anti-inverse subsemigroups.

The converse follows immediately by Theorem 3.4.

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