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ON PARTIALLY DIRECTED EULERIAN MULTIGRAPHS

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Abstract: **In the paper two necessary and sufficient conditions for partially directed multigraph to be Eulerian are presented.**

In this paper we consider only finite partially directed graphs that can have loops and multiple lines.

Graph G is ordered triple (V, E, A), where V, E and A are finite pairwise disjoint sets. V is the set of *vertices*. E is the set of undirected lines, called *edges*. To each edge $e \in E$ we associate an unordered pair $[u, v]$ of vertices, $u, v \in V$, that are called *endpoints* of e. If $u = v$ then e is undirected loop. A is the set of directed lines, called *arcs*. To each arc $a \in A$ there is associated an ordered pair (u, v) of vertices. u is *tail* and v is *head* of a. If $u = v$ then a is directed loop.

Graph $G = (V, E, A)$ is *directed* if $E = \varnothing$. It is undirected if $A = \varnothing$. Let G be any graph. Graphs $E(G) = (V, E, \varnothing)$ and $A(G) = (V, \varnothing, A)$ are *unoriented* and *oriented* part of G respectively.

If we substitute in a graph G an arc a whose pair is (u, v) by new edge $e(a)$ with corresponding pair $[u, v]$ we say that $e(a)$ is *desorientation* of a. The reverse operation that is in general not unique is called *orientation* $a(e)$ of edge e. G is an *orientation* (*desorientation*) of graph H if it is obtained by successive orientations (desorientations) of edges (arcs) of H. An orientation (desorientation) is *complete* if the resulting graph is directed (undirected).

Path g in a graph G is any sequence of vertices $v_0, \ldots, v_n$ and lines $p_0, \ldots, p_{n-1}$ of the form:

$$g = v_0 \, p_0 \, v_1 \cdots v_{n-1} \, p_{n-1} \, v_n$$

where all lines p_t are mutually distinct and if p_i is directed line ($=$arc) then v_i is tail and v_{i+1} is its head and if p_i is undirected line ($=$edge) then v_i and v_{i+1} are its endpoints.

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We say that path g has length n and that is leading *from* v_0 *to* v_n. If $v_0 = v_n$ then g is a *cycle*. Note that any edge p in a path g can be oriented in unique way so that g remains a path. We say that p is oriented *in accordance* with g. Any orientation of edges of a path g in accordance with g is called an *orientation* of path g. If g has no edges it is an *oriented* path if it has no arcs it is *unoriented* path.

Two paths g and h with no lines in common are said to be *disjoint*. The set of pairwise disjoint cycles of graph G that covers all lines of G is called *cycle decomposition* of G. A cycle decomposition is *minimal* if there is no cycle decomposition with fewer cycles. Graph G is *Eulerian* if there is a cycle (called Eulerian) that covers all vertices and lines of G.

Let u and v be any vertices in G. They are *strongly connected* if there is a path leading from u to v and path from v to u. They are said to be *one way connected* if there exists a path that is leading from u to v or from v to u. They are *weakly connected* if they are (strongly) connected in a graph H that is complete desorientation of G. It is clear that vertices that are strongly connected are also one way connected and if they are one way connected then they are also weakly connected. So we say that u and v are *not connected* if they are not even weakly connected. G itself is strongly (one way, weakly) connected if any pair of vertices of G is strongly (one way, weakly) connected. G is disconnected if it is not even weakly connected. In undirected graphs all three types of connectivity coincide. In that what follows we will use the term *connected* as weakly connected. The following well known theorem allows us to study cycle decompositions of graphs instead of Eulerian graphs.

Theorem 1: *G is Eulerian if and only if it is connected and has a cycle decomposition.*

Let u and v be any two vertices in G. By $a(u, v)$ we denote the number (multiplicity) of arcs in G with tail u and head v. By $e(u, v)$ we denote the number of edges with endpoints u and v. If X and Y are any subsets of V and $\overline{X} = V - X$ we use the following notations:

$$a(X, Y) = \sum_{u \in X,\, v \in Y} a(u, v)$$

$$e(X, Y) = \sum_{u \in X,\, v \in Y} e(u, v)$$

$$e(X) = e(X, \overline{X})$$

$$a^+(X) = a(\overline{X}, X)$$

$$a^-(X) = a(X, \overline{X})$$

$$c(X, Y) = a(Y, X) - a(X, Y)$$

$$c(X) = c(X, \overline{X}) = a^+(X) - a^-(X)$$

$c(X)$ will be called the *charge* of X. ∂X or *boundary* of X consists of all lines having one endpoint in X and the other one in $\overline{X}$. The number of lines in ∂X is *valency* of X denoted as val (X). Here are some relations that will be used later on.

$$e(X)=e(\overline{X}), \quad a^+(X)=a^-(\overline{X}), \quad \mathrm{val}(X)=e(X)+a^+(X)+a^-(X)$$

$$c(X, Y)=-c(Y, X), \quad c(X)=-c(\overline{X}), \quad c(X, X)=0$$

Lemma 1: *Charge is an additive function; i.e:*

$$\text{for all} \quad X\subset V: \quad c(X)=\sum_{v\in X} c(v)$$

Proof: Let $t\in X$ and $Y=X-t$. Then:

$$\begin{aligned} c(X) &= \sum_{v\in X,\, u\in\overline{X}} c(v, u) = \sum_{v\in Y,\, u\in\overline{X}} c(v, u) + \sum_{u\in\overline{X}} c(t, u) = \\ &= \sum_{v\in Y,\, u\in\overline{Y}} c(v, u) - \sum_{v\in Y} c(v, t) + \sum_{u\in\overline{X}} c(t, u) = \\ &= c(Y) + \sum_{v\in V} c(v, t) = c(Y)+c(t) \end{aligned}$$

Thus: $c(X)=c(X-t)+c(t)$. From this recursive relation the additivity of c follows.

If $c(X)$ equals zero (is positive, negative) we say that X is *neutral* (*positive, negative*). Graph G is *neutral,* if all $X\subset V$ are neutral. Note that G is neutral if and only if all vertices are neutral.

We say that X is *even* (*odd*) if it has even (odd) valency. G itself is *even* if all its vertices are even.

Using the terminology developed so far we are able to paraphrase the wellknown results concerning directed and undirected graphs [1, 3, 5]:

Theorem 2: *Directed graph admits a cycle decomposition if and only if it is neutral.*

Theorem 3: *Undirected graph admits a cycle decomposition if and only if it is even.*

The only attempt known to the authors, to solve the problem for general graphs (i.e. partially directed multigraphs) is published in [2], pp 419 – 421 as theorem XII — 5 which, transcribed in to our terminology says:

Hypothesis: Let G be strongly connected graph. The necessary and sufficient condition for G to possess an Eulerian cycle is: for any vertex $v \in V$, $e(v) - |c(v)|$ must be nonnegative even integer.

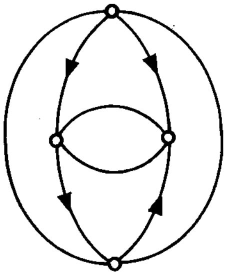

Fig. 1

It is almost evident that the condition of the hypothesis is necessary. Unfortunately the graph in Fig. 1 fulfills it, yet it is not Eulerian.

Let X be a vertex set having a charge $c(X)$. For X to become neutral, one should orient $|c(X)|$ edges of ∂X in a prescribed way. We introduce

$$P(X) = e(X) - |c(X)|.$$

If $P(X) < 0$, no orientation of G can render X neutral.

Let us see what happens to $P(X)$ upon orientation or desorientation of some line p. Clearly $P(X)$ does not change if we orient or desorient lines that do not belong to ∂X. When desorienting an arc from ∂X, $c(X)$ will be changed by 1 and $e(X)$ will be increased by 1. Thus $P(X)$ either remains the same or it is increased by two. This implies that an orientation of an edge from ∂X either has no effect on $P(X)$ or diminishes it by 2. In fact if one of the two possible orientations diminishes $P(X)$ the other will have no effect on it and vice versa. Thus we proved:

Lemma 2: *Let p be any line and X any vertex set of G. An orientation (desorientation) of p*

a) *has no effect on $P(X)$ if $p \notin \partial X$*

b) *does not change $P(X)$ or decreases (increases) it by 2 if $p \in \partial X$*

Now we are ready to prove the theorem that solves the problem of Eulerian cycles in general graphs.

Theorem 4: *Let G be any graph. G has a cycle decomposition if and only if for each vertex set $X \subset V$, $P(X)$ is nonnegative even integer.*

Proof: The "if" part: The cycles of any cycle decomposition of G induce a complete orientation H of G. Following theorem 2 in H each vertex is neutral: $c_H(v) = 0$. We shall use subscript to indicate the graph. Whenever the subscript is omitted the reference graph is G. Since H is directed graph: $P_H(X) = -|c_H(X)|$. By lemma 1 c is additive and $c_H(X)$ must be zero. We proved, that $P_H(X) = 0$ for all $X \subset V$. Graph G is a desorientation of H. By lemma 2 $P(X) \geqslant P_H(X)$ and $P(X) - P_H(X)$ is even. So $P(X)$ must be nonnegative even integer.

To prove the "only if" part of the theorem we prove that any edge can be oriented in such a way as to preserve the condition of the theorem. Next we can stepwise orient the edges until we reach oriented graph. Since the condition of this theorem implies the condition of theorem 2 in case of oriented graphs there exists a complete orientation of G that admits a cycle decomposition. Hence G itself admits a cycle decomposition.

Let e be any edge of G. We are going to prove that e can be oriented so that the conditions of the theorem remain valid on every set X. There are two cases:

1. All vertex sets $X \subset V$, such that $e \in \partial X$, have the property $P(X) > 0$.
2. There is some $X \subset V$, such that $e \in \partial X$ and $P(X) = 0$.

In the case no. 1 any orientation of e will preserve the condition. This is ensured by lemma 2.

The case no. 2 is much harder. To preserve the condition on X, e must be oriented in a prescribed way — the other orientation would necessarly destroy the condition, again by lemma 2. We have to prove that this forced orientation will not cause the violation of the condition $P(Y) \geqslant 0$ for some other vertex set $Y \subset V$. This could only happen if $e \in \partial Y$ and $P(Y) = 0$.

V can be viewed as a disjoint union of four sets A, B, C and D:

$$A = X \cap Y, \quad B = X - A, \quad C = Y - A, \quad D = V - (X \cup Y).$$

Since the conditions of the theorem concern all vertex sets, we have:

(1) $P(A) = 2\,m_A$ (2) $P(B) = 2\,m_B$

(3) $P(C) = 2\,m_C$ (4) $P(D) = 2\,m_D$

(5) $P(X) = 0$ (6) $P(Y) = 0$

where $m_A, m_B, m_C, m_D \geqslant 0$. We shall expand this system of identities substituting $c(A)$ by $c(A, B) + c(A, C) + c(A, D)$, $e(A)$ by $e(A, B) + e(A, C) + e(A, D)$ and so on. To shorten the notation we are going to use subscripts according to fig. 2. instead of ordered pairs, i.e,: $e_a = e(B, A) = e(A, B)$, $c_a = c(B, A) = -c(A, B)$ etc. Note that labelling and orientations of graph in fig. 2 is arbitrarily chosen. The arrows indicate the order of pairs.

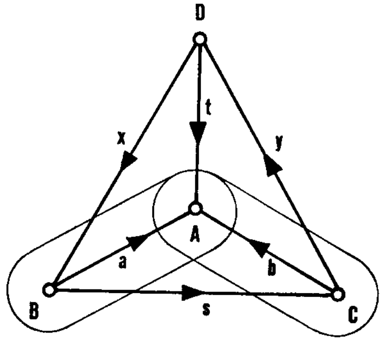

Fig. 2

The equations (1) — (6) are now rewritten as follows:

(1′) $$e_a + e_b + e_t = 2\,m_A + |c_a + c_b + c_t|$$

(2′) $$e_x + e_a + e_s = 2\,m_B + |c_x - c_a - c_s|$$

(3′) $$e_s + e_b + e_y = 2\,m_C + |c_s - c_b - c_y|$$

(4′) $$e_y + e_t + e_x = 2\,m_D + |c_y - c_t - c_x|$$

(5′) $$e_x + e_t + e_b + e_s = |c_x + c_t + c_b - c_s|$$

(6′) $$e_a + e_s + e_t + e_y = |c_a + c_s + c_t - c_y|$$

We continue the proof by reductio ad absurdum.

Suppose an orientation of e either violates the condition on X or on Y. Then e has one of its endpoints in B and the other in C (it belongs to *stream* $s = (B, C)$) or e belongs to stream $t = (D, A)$

Case A: e belongs to the stream s.

$c(X)$ and $c(Y)$ are either both positive or both negative. This implies:

$$|c(X)|+|c(Y)|=|c(X)+c(Y)| \tag{7a}$$

Using (7a) and the triangular inequality for absolute values we add (5′) and (6′) and we have:

$$e_x+e_y+2(e_s+e_t)+e_a+e_b=|c_x+2c_t+c_a+c_b-c_y|\leqslant$$
$$\leqslant|c_a+c_b+c_t|+|c_t+c_x-c_y|=|c(A)|+|c(D)| \tag{8a}$$

Substituting $c(A)$ and $c(D)$ in (8a) by those from (1′) and (4′) we get:

$$e_x+e_y+2(e_s+e_t)+e_a+e_b\leqslant e_x+e_y+2e_t+e_a+e_b-2(m_A+m_D)$$

this yielding:

$$e_s\leqslant-(m_A+m_D)\leqslant 0 \quad \text{and finally} \quad e_s=0.$$

But $e_s=0$ contradicts our assumption that e belongs to stream s.

Case B: e belongs to the stream t. (thus $e_t>0$)

$c(X)$ and $c(Y)$ are neither both positive nor both negative. This implies

$$|c(X)|+|c(Y)|=|c(X)-c(Y)| \tag{7b}$$

By the same argument as in case A we get:

$$e_x+e_y+2(e_s+e_t)+e_a+e_b\leqslant|c(B)|+|c(C)| \tag{8b}$$

Substituting $c(B)$ and $c(C)$ in (8b) by those from (2′) and (3′) we deduce $e_t\leqslant-(m_B+m_C)\leqslant 0$ and finally $e_t=0$. This is impossible since $e_t>0$.

We proved that any edge $e\in\partial X\cap\partial Y$ with $P(X)=P(Y)=0$ can be oriented so that neither the condition on X nor that on Y are violated. This completes the proof of the theorem.

Lemma 3: *In an even graph, for all $X\subset V$, $P(X)$ is even.*

Proof: By lemma 2 the parity of $P(X)$ does not change upon an orientation of graph. Since any orientation of even graph is even it remains to prove this lemma only for directed graphs. In a directed even graph val $(v)=a^+(v)+a^-(v)$ is even for all vertices $v\in V$. It follows that $c(v)=a^+(v)-a^-(v)$ is even to. Since $c(X)$ is additive, $c(X)$ is even for all $X\subset V$. Finally $P(X)=-|c(X)|$ must be even.

If we put together theorem 1, theorem 4 and lemma 3 we observe:

Corollary 1: *A graph G is Eulerian if and only if it is connected, even and for all $X\subset V$, $P(X)\geqslant 0$.*

Unfortunately in general the straigth forward application of this corollary for testing whether a graph is Eulerian is highly impractical since there are $2^{|V|}$ sets X to be tested against $P(X)<0$. To get better methods for testing whether some vertex set X has the property $P(X)<0$ we have to develop some more theories.

Let u be a positive and v a negative vertex in G. A path g leading from u to v and lying entirely on $E(G)$ will be called a *discharging path*. A set of pairwise disjoint discharging paths with the proprety that a vertex w is an endpoint of at most $c(w)$ discharging paths will be called a *discharging of a graph*. $d(G)$ will denote the maximum cardinality of all dischargings of G. By $C(G)$ we will denote $(1/2)\sum_{v\in V}|c(v)|$.

We say that $C(G)$ is a *charge* of G. Note that G is neutral if and only if $C(G)=0$. $C(G)$ is also the sum of charges of all positive vertices. Orientation of a discharging path from u to v only affects charges on u and v. It decreases $c(u)$ by 1 and increases $c(v)$ by 1. Thus it decreases $C(G)$ by exactly 1. Orientation of discharging paths of a maximal discharging will reduce $C(G)$ by $d(G)$. Thus $d(G)\leqslant C(G)$. If $d(G)=C(G)$, we say that G is dischargeable.

Theorem 5: *G is dischargeable if and only if there is no vertex set X in G with the property that* $P(X)<0$.

Proof: If G is dischargeable it can be oriented into a graph H so that $C(H)=0$. Thus each vertex set X is neutral in H and

$$P_H(X)=e_H(X)-|c_H(X)|=e_H(X)\geqslant 0$$

But any desorientation of H only increases P, so $P(X)\geqslant 0$ for all X.

We use the contradiction to demonstrate the second implication. Assume $P(X)\geqslant 0$ for all X and $d(G)<C(G)$. We are going to construct an undirected graph G' starting with $E(G)$. Introduce new vertices s and t. Vertex s is connected to each positive vertex u by $c(u)$ edges. Vertex t is connected to each negative vertex v by $-c(v)$ edges. Graph G' is so constructed. The paths from s to t will be called (as usually) *$s-t$ paths.* To each discharging path there is associated at least one $s-t$ path. To each discharging of G we can associate a set of pairwise disjoint $s-t$ paths. Even more: the maximal number of disjoint $s-t$ paths equals $d(G)$! By theorem 5.11 in [3] (a variant of Menger theorem) the maximum number of disjoint $s-t$ paths equals the minimum edge cut separating s and t. Let $\partial X'$ be minimum cut such that $s\in X'$ and $t\in\overline{X}'$. See fig. 3.

We use the following notations:

$$X=X'-s,\quad \overline{X}=\overline{X}'-t=V-X,$$

$$k=e_{G'}(s,X),\quad m=e_{G'}(t,X),\quad n=e_{G'}(X,\overline{X})$$

Let $c=C(G)$ and $d=d(G)$. Then

$$d=c-k+n+m$$

By hypothesis we have $d<c$. Thus:

$$n-|k-m|<0$$

But $c(X)=e_{G'}(s, X)-e_{G'}(t, X)$ and $e(X)=e_{G'}(X, \bar{X})$ then:

$$P(X)=e(X)-|c(X)|=n-|k-m|<0$$

This contradicts the hypothesis $P(X)\geqslant 0$ for all X. The theorem is proved.

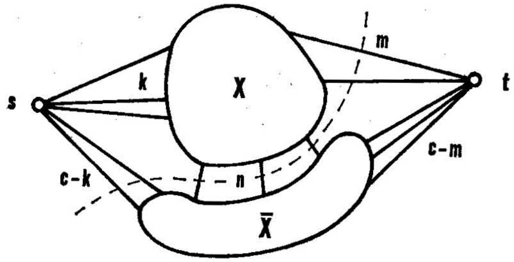

Fig. 3

If we want to traverse Eulerian cycles in Eulerian graph we need an algorithm to test if graph is Eulerian, to construct Eulerian cycles or to explain why graph is not Eulerian. So it is better to turn corollary 1 around:

Corollary 2: *Given any graph G. It belongs to exactly one of the following categories*:

C_1: *G has some vertices of odd valency.*

C_2: *G is even but there exists* $X\subset V$ *with* $P(X)<0$.

C_3: *G is disconnected. Each component is Eulerian.*

C_4: *G is Eulerian graph.*

An algorithm to classify graphs according to corollary 2 can be constructed along the following lines:

A. {Test for odd vertices} For each vertex compute its valency. If there is any odd vertex present, return "odd vertices" and stop;

B. {Test for the set X with $P(X)<0$} Find maximal discharging and label appropriately vertices. The set X consists of all labelled vertices. If (at the end of this step) X is not empty, return "X with $P(X)<0$" and stop, otherwise orient all edges of G lying on discharging paths. (Note: this step consists of slight modification of Ford-Fulkerson Max-flow algorithm — see [4]);

C. {Cycle decomposition} Construct a minimal cycle decomposition of G. Number of cycles is the number of graph components. If G is not connected, return "Eulerian components" otherwise return "Eulerian graph" and stop.

A Fortran implementation of the described algorithm can be obtained from the authors.

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