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NONOSCILLATION OF NONLINEAR RETARDED DIFFERENTIAL EQUATIONS

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Abstract. **The forced n-th orber retarded equation**

$$y^{(n)}(t)-\sum_{i=1}^{m} p_i(t)\, h_i\big(y(g_i(t))\big)=f(t)$$

is studied for its nonoscillatory behavior, where the nonnegativity of the function p_i is not assumed, for each $i=1, 2, \ldots, m$. Under sufficient hypotheses, it is shown that if $y(t)$ is a bounded solution, then $y(t)$ is oscillatory er tending to zero as $t\to\infty$.

Recently, B. Singh [3] has considered the forced fourth order linear retarded equation

$$y^{(4)}(t)+a(t)\,y(g(t))=b(t), \quad 0<g(t)\leqslant t, \tag{1}$$

where $a(t), g(t), b(t)\in C[R_+, R]$, $R_+=[0, \infty)$, $R(-\infty, \infty)$. And under some sufficient conditions, he proved that every bounded solution of (1) tends to zero as $t\to\infty$.

The purpose of this paper is to discuss the corresponding result of Singh's to the following nonlinear equation with retarded arguments

$$y^{(n)}(t)-\sum_{i=1}^{m} p_i(t)\, h_i(y[g_i(t)])=f(t) \tag{2}$$

by using Kolmogorov's general theorem [2].

Throughout this paper the following assumptions are assumed to hold:

(a) $p_i\in C[R_+, R]$, $i=1, 2, \ldots, m$,

(b) $g_i\in C[R_+, R_+]$, $g_i(t)\leqslant t$, $\lim\limits_{t\to\infty} g_i(t)=\infty$, $g_i(t)$ is differentiable and has bounded derivative on R_+, $i=1, 2, \ldots, m$,

(c) $f\in C[R_+, R]$, $\int^{\infty} f(t)\,dt<\infty$,

(d) $h_i \in C[R, R]$, $xh_i(x)>0$ for $x\neq 0$, and there exist two positive constants K and M such that

$$K\leqslant \frac{h_i(x)}{x}\leqslant M, \quad x\neq 0,$$

for every $i=1, 2, \ldots, m$.

Sufficient smoothness for the existence of solutions for all large t will be assumed without mention. In what follows we consider only such solutions which are nontrivial for all large t. The oscillatory character is considered in the usual sense, i.e, a continuous real-valued function which is defined for all large t is called oscillatory if it has no last zero, otherwise it is called nonoscillatory.

Theorem. *Let the conditions (a)–(d) hold. Suppose that there exist two positive constants k and ε such that*

$$\text{(A)} \qquad \liminf_{t\to\infty} \sum_{i=1}^{m} \int_t^{t+k} p_i^+(s)\,ds \geqslant \varepsilon > 0, \quad p_i^+(t)=\max\{p_i(t),\ 0\},$$

$$\sum_{i=1}^{m} \int^{\infty} p_i^-(t)\,dt < \infty, \quad p_i^-(t)=\max\{-p_i(t),\ 0\}.$$

Then every bounded solution $y(t)$ of (2) is oscillatory or tending to zero as $t\to\infty$.

Proof. Let $y(t)$ be nonoscillatory. Without any loss of generality, we may assume that $y(t)$ is positive eventually. Since $y(t)$ is bounded, there is a positive constant L such that for $t>0$

$$|y(t)|\leqslant L. \tag{3}$$

Let T be large enough so that $y(t)$ and $y(g_i(t))$, $i=1, 2, \ldots, m$, are positive for $t\geqslant T$, Integrating both sides of (2) from T to t, we obtain

$$y^{(n-1)}(t)-y^{(n-1)}(T)-\sum_{i=1}^{m}\int_T^t p_i^+(s)\,h(y[g_i(s)])\,ds = -\sum_{i=1}^{m}\int_T^t p_i^-(s)\,h(y[g_i(s)])\,ds+\int_T^t f(s)\,ds. \tag{4}$$

From (3), (4) and (d) we have

$$y^{(n-1)}(t)-y^{(n-1)}(T)-M\sum_{i=1}^{m}\int_T^t p_i^+(s)\,y(g_i(s))\,ds \leqslant LK\sum_{i=1}^{m}\int_T^t p_i^-(s)\,ds+\int_T^t f(s)\,ds. \tag{5}$$

Equation (5) suggests that

$$\sum_{i=1}^{m} \int_{T}^{\infty} p_i^+(s)\, y(g_i(s))\, ds < \infty. \tag{6}$$

In fact, if

$$\sum_{i=1}^{m} \int_{T}^{\infty} p_i^+(s)\, y(g_i(s))\, ds = \infty,$$

then due to condition (c), it follows from (5) that $y^{(n-1)}(t) \to \infty$ as $t \to \infty$. But this will force $y(t)$ to be unbounded, a contradiction. Hence (6) holds.

From condition (A) we have

$$\sum_{i=1}^{m} \int^{\infty} p_i^+(t)\, dt = \infty. \tag{7}$$

From (6) and (7) we obtain

$$\liminf_{t\to\infty} y(t) = \liminf_{t\to\infty} y(g(t)) = 0, \tag{8}$$

where the subscript i in $g_i(t)$ is dropped for convenience.

Now we will prove that

$$\lim_{t\to\infty} y'(t) = 0. \tag{9}$$

From (5) and (6), we see easily that

$$y^{(n-1)}(t) \to 0$$

as $t \to \infty$. Hence for any small $\varepsilon_1 > 0$, there exists a $t_1 \geqslant T$ such that for $t \geqslant t_1$

$$|y^{(n-1)}(t)| \leqslant \varepsilon_1. \tag{10}$$

Now we make use of Kolmogrov's general theorem that, if $|y(t)| \leqslant M_0$ and $|y^{(n-1)}(t)| \leqslant M_{n-1}$ on R_+, then

$$|y^{(i)}(t)| \leqslant k_{n,i}\, M_0^{1-\frac{i}{n-1}}\, M_{n-1}^{\frac{i}{n-1}},$$

where $k_{n,i}$ is a positive constant depending on n, i and $0 < i < n-1$. (See [2, p. 22]). It follows from (3) and (10) that

$$y^{(i)}(t) \to 0, \quad i = 1, 2, \ldots, n-2,$$

as $t \to \infty$. Hence (9) holds.

Suppose

$$\limsup_{t\to\infty} y(t) > r > 0. \tag{11}$$

From (8) and (11), there exists a sequence $\{b_j\}$, $j \geqslant 0$ with the following properties (see [1])

(c_1) $\lim_{j\to\infty} b_j = \infty$, $b_j \geqslant t_1$ for all j, t_1 is the same as above.

(c_2) For each j, $y(g(b_j)) > r$.

(c_3) For each $j \geqslant 1$, there exist numbers b_j' such that $b_{j-1} < b_j' < b_j$ and $y(g(b_j')) < \frac{r}{2}$.

Let a_j be the largest number less than b_j such that $y(g(a_j)) = \frac{r}{2}$ and c_j be the smallest number greater than b_j such that for $j \geqslant 1$.

$$y(g(c_j)) = \frac{r}{2}. \tag{12}$$

Now in the interval $[a_j, b_j]$, there exists a d_j such that by mean value theorem

$$g'(d_j)\, y'(g(d_j)) = \frac{y(g(b_j)) - y(g(a_j))}{b_j - a_j} > \frac{r}{2(c_j - a_j)}. \tag{13}$$

But (9) implies that $y'(g(d_j)) \to 0$ as $g(d_j) \to \infty$. Condition (b) implies $g'(d_j)$ is bounded. Therefore it follows from (13) that

$$\lim_{j\to\infty} (c_j - a_j) = \infty.$$

Also because of the way a_j and c_j were chosen

$$y(g(t)) \geqslant \frac{r}{2} > 0$$

on $[a_j, c_j]$. Now from (6), it follows that

$$\infty > \sum_{i=1}^{m} \int_{t_1}^{\infty} p_i^+(s)\, y(g_i(s))\, ds \geqslant \sum_{i=1}^{m} \left[\sum_{j=1}^{\infty} \int_{a_j}^{c_j} p_i^+(s)\, y(g_i(s))\, ds \right]$$

$$> \frac{r}{2} \sum_{i=1}^{m} \left[\sum_{j=1}^{\infty} \int_{a_j}^{c_j} p_i^+(s)\, ds \right] = \infty,$$

due to condition (A). This contradiction shows that $r = 0$ and our proof is complete.

Example. Consider the following equation

$$y^{(6)}(t) - e^{\frac{t}{2} - \pi}\, y(t - \pi) - e^{-\pi}\, y\left(\frac{t}{2} - \pi\right) = 2\, e^{-\frac{t}{2}} - e^{-t},\ t > 2\pi$$

which has $y(t) = -e^{-t}$ as a bounded nonoscillatory solution that goes to zero as $t \to \infty$. Here

$$f(t) = 2e^{-\frac{t}{2}} - e^{-t}, \qquad \int^{\infty} f(t)dt < \infty,$$

$$p_1^+(t) + p_2^+(t) = e^{\frac{t}{2} - \pi} + e^{-\pi}, \qquad \int^{\infty} (p_1^+(t) + p_2^+(t))\, dt = \infty,$$

$$p_1^-(t) = p_2^-(t) = 0.$$

Thus all the conditions of Theorem 2 are satisfied.

Corollary. *Let the conditions* $(a)-(d)$ *and* (A) *hold. if*

$$p_i(t) \geqslant 0, \quad i = 1, 2, \ldots, m$$

then every bounded solution of equation (2) *oscillates or tends to zero as* $t \to \infty$.

REFERENCES

[1] M. E. Hammett, *Nonoscillation properties of a nonlinear differential equation*, Proc. Amer. Math. Soc., 30 (1971), 92—96.

[2] I. J. Schoenberg, *The elementary cases of Landau's problem of inequalities between derivatives*, MRA Technical Summary Report # 1147, Feb. (1972), Mathematics Research Center, The University of Wisconsin, Madison. (Reproduced in Amer. Math. Monthly, 80 (1973), 121—158).

[3] B. Singh, *Nonscillation of forced fourth order retarded equations*, SIAM J, Appl. Math. 28 (1975), 265—269.

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