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CHARACTERISTIC MATRIX FUNCTIONS ON WACHS SPACES, I (LIVŠITS DEFINITION)

Aleksandar Torgašev

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1.

The Livšits papers [1], [2] were the first papers where the characteristic matrix function (briefly the c.m.f.) of an isometric operator (or of a general bounded linear operator) with finite deficiency indices was considered.

Later, this concept was developed and transferred to some classes of general bounded linear operators on the complex Hilbert spaces by V. M. Brodskiĭ, A. V. Štrauss, C. Foiaş and B. Sz. Nagy and others.

In this paper we consider the characteristic matrix function of an isometric operator on Wachs spaces, i.e. on the left quaternionic Hilbert spaces, with quaternionic bilinear form $\langle x, y\rangle$, observing that all obtained results are closely related to the (we think we can say) classical results of M. S. Livšits [2].

Since every Wachs space H is a special complex Hilbert space with the form $[x, y] =$ compl. part $\langle x, y\rangle$ $(x, y \in H)$, certain constructions and properties are (easily or hardly) translated to our situation, but in view of the existence of the specific quaternionic structure, sometimes some special properties of c.m.f. arise.

We consider here mainly such properties (characteristic properties) of the c.m.f. of an isometric operator, by which they differ from the ordinary c.m.f. of operators on the corresponding complex Hilbert space H^s.

Throughout the paper, H is a left Wachs space, H^s-the corresponding complex Hilbert space, and $B(H)$, $B(H^s)$ the algebras of all bounded linear operators on H and H^s respectively. Since we assume that the field C of complex numbers is imbedded in the noncommutative field of quaternions Q, obviously $B(H)\subseteq B(H^s)$.

We often denote by an index "s" notions corresponding (in H^s) to notions in H.

2.

Let V be an isometric operator in H, with the closed domain $D(V)$ $(D(V) \neq H)$, with the closed range $R(V)$ $(R(V) \neq H)$ - both subspaces in H, and let us suppose

$$\dim D(V)^{\perp} = \dim R(V)^{\perp} = n < \infty.$$

Then we say that V has deficiency indices (n, n).

It immediately follows that deficiency indices of V in H^s are $(2n, 2n)$.

Denoting by $O(V)$ and $O^s(V)$ respectively the sets of all bounded orthogonal extensions of V on H (i.e. H^s), we have $O(V) = O^s(V) \cap B(H)$.

For subspaces

$$G_z = R(I - \bar{z}V), \quad G'_z = R(V - zI) \qquad (|z| \neq 1)$$

in H^s, we have that

$$\dim_s (G_z)^{\perp_s} = \dim_s (G'_z)^{\perp_s} = 2n$$

([2], p. 249).

We remark that in general case $(|z| \neq 1, z \notin R)$, the closed subspaces G_z, G'_z (in H^s) are not subspaces in H.

By the way, we point out the following relations.

(1) For every $z \in C$, $G_{\bar{z}} = KG_z$ and $G'_{\bar{z}} = KG'_z$ hold (where $Kx = jx$, $x \in H$), thus G_z, G'_z are subspaces of H at least for real values z.

(2) For every z $(|z| \neq 1, z \notin R)$, $G'_z + G'_{\bar{z}} \supseteq D(V)$.

The proof of (1) is immediate.

Since $y \in (G'_z)^{\perp}$ is equivalent to $\langle Vx, y \rangle = z \langle x, y \rangle$ $(x \in D(V))$, interstanding x by Kx we find that

$$\langle V(Kx), y \rangle = z \langle Kx, y \rangle = zj \langle x, y \rangle$$
$$= j \langle Vx, y \rangle = jz \langle x, y \rangle \qquad (\forall x \in D(V))$$

thus $z = \bar{z}$ or $y \in D(V)^{\perp}$.

Since by assumption $z \notin R$, we get $(G'_z)^{\perp} \subseteq D(V)^{\perp}$, so that

$$(G'_z)^{\perp\perp} \supseteq D(V)^{\perp\perp} = \overline{D(V)} = D(V).$$

But since $(G'_z)^{\perp\perp}$ is the minimal closed subspace (in H) containing G'_z, thus exactly $G'_z + KG'_z = G'_z + G'_{\bar{z}}$ (which is closed because G'_z, $G'_{\bar{z}}$ are closed for $|z| \neq 1$), we get

$$G'_z + G'_{\bar{z}} \supseteq D(V),$$

which was asserted. □

If T is an arbitrary extension from $O(V)$, and $K_T=\{z:|z|\leqslant\|T\|^{-1}\}$, we have

$$(2.1)\qquad \begin{cases}(G_z)^{\perp_s}=(I-zT^*)^{-1}D(V)^{\perp}\\ (G'_z)^{\perp_s}=(I-\bar{z}T)^{-1}R(V)^{\perp}\end{cases},$$

for every $z\in K_T$ ([2], p. 250).

3.

Since the dimensions of subspaces $D(V)^{\perp}$, $R(V)^{\perp}$ (in H) are equal to n, there exist two orthonormal bases - $g_1, \dots, g_n$ in $D(V)^{\perp}$ and $g'_1, \dots, g'_n$ in $R(V)^{\perp}$, from which using the relation

$$\langle x, y\rangle=[x, y]-j[Kx, y] \qquad (x, y\in H)$$

we conclude that $\{g_1, \dots, g_n, Kg_1, \dots, Kg_n\}$ and $\{g'_1, \dots, g'_n, Kg'_1, \dots, Kg'_n\}$ are complex o.n.b. in these subspaces; thus

$$[f_p, f_q]=[f'_p, f'_q]=\delta_{p,q} \qquad (p, q=1, \dots, 2n),$$

where

$$f_p=\begin{cases} g_p, & 1\leqslant p\leqslant n\\ Kg_{p-n}, & n+1\leqslant p\leqslant 2n\end{cases}, \qquad f'_p=\begin{cases} g'_p, & 1\leqslant p\leqslant n\\ Kg'_{p-1}, & n+1\leqslant p\leqslant 2n\end{cases}.$$

We say that such bases are regular, and in further text we consider only the regular bases in these subspaces.

Further, for an arbitrary $T\in O(V)$, we according to [2] construct the vectors

$$f_p(z)=(I-zT^*)f_p, \qquad f'_q(z)=(I-\bar{z}T)^{-1}f'_q$$

$(p, q=1, \dots, 2n;\ z\in K_T)$, and the characteristic matrix function of Livšits (of the operator T) is defined by

$$(3.1)\qquad \Omega_T=(z)=\{\omega_{p,q}(T; z)\}\in M_{2n}(C)$$

($M_{2n}(C)$ — the set of all complex square matrices of order $2n$), where

$$\omega_{p,q}(T; z)=[(T-zI)f_p(z), f'_q]$$

$(p, q=1, \dots, 2n)$.

Denote by $P_{2n}(C)=P_{2n}$ the set of all square $2n\times 2n$ complex matrices of the form

$$\begin{bmatrix} S_1 & -\bar{S}_2\\ S_2 & \bar{S}_1\end{bmatrix}$$

where $S_1, S_2\in M_n(C)$.

L e m m a 1. *Let* $\tau=\{\tau_{p,q}\}\in M_{2n}(C)$ *be the "matrix" of* T *defined by*

$$Tf_p=\sum_{q=1}^{2n}\tau_{p,q}f'_q \qquad (p=1,\dots,2n),$$

i.e. $\tau_{p,q}=[Tf_p, f'_q]$. *Then* $\tau\in P_{2n}$ *and the relation* $\tau\in P_{2n}$ *is characteristic for extensions* $T\in O(V)$ (*with respect to the extensions from* $O^s(V)$).

We omit the proof. □

Defining next $G_T(z)=\{[f_p(z), f_q]\}_{2n\times 2n}$, $G'_T(z)=\{[f_p(z), f'_q]\}_{2n\times 2n}$ $(z\in K_T)$, the Livšits characteristic function of T is

$$\Omega_T(z)=[G_T(z)-zG'_T(z)\tau^*]^{-1}[G_T(z)-zG'_T(z)] \tag{3.2}$$

and the matrix function of V is

$$\Omega(z)=\Omega_V(z)=-zG(z)^{-1}G'(z) \qquad (|z|\leqslant 1). \tag{3.3}$$

It holds

$$\Omega_T(z)=[I_{2n}+\Omega(z)\tau^*]^{-1}[\tau+\Omega(z)]. \tag{3.4}$$

For an arbitrary $T\in O(V)$, we denote by $Q_{2n}(T)$ the set of all matrix functions $S(z)\in M_{2n}(C)$ $(z\in K_T)$ of the form

$$\begin{bmatrix} S_1(z) & -\overline{S_2(\bar z)} \\ S_2(z) & \overline{S_1(\bar z)} \end{bmatrix}, \tag{*}$$

where $S_1(z)$, $S_2(z)\in M_n(C)$ are $n\times n$ matrix functions defined on the set $K_T=\{z\,\|\,z|\leqslant 1/\|T\|\}$.

Obviously, if $S(z)\in Q_{2n}(T)$ then for every real r $(r\in K_T)$ matrix $S(r)\in P_{2n}$.

L e m m a 2. *The product of two matrices* $S(z)$, $R(z)$ *of the class* $Q_{2n}(T)$ *is a matrix of this class too.*

If for an $S(z)\in Q_{2n}(T)$, *the inverse matrix* $S(z)^{-1}$ *there exists for every* $z\in K_T$, *then* $S(z)^{-1}\in Q_{2n}(T)$.

We omit the proofs. □

T h e o r e m 1. *For every* $T\in O(V)$ *the matrix function* $\Omega_T(z)\in Q_{2n}(T)$. *If conversely* $\Omega_T(z)\in Q_{2n}(T)$ *for a* $T\in O^s(V)$ *then* $T\in O(V)$.

P r o o f. We prove first that matrices $G_T(z)$, $G'_T(z)$ belong to $Q_{2n}(T)$ and then apply the previous lemma.

We have for every $z\in K_T$ and $p, q=1,\dots,n$:

$$\begin{aligned}[f_{n+p}(\bar z), f_{n+q}]&=[(I-\bar zT^*)^{-1}f_{n+p}, f_{n+q}]=\\&=[(I-\bar zT^*)^{-1}Kg_p, Kg_q]=[K(I-zT^*)^{-1}g_p, Kg_q]=\\&=\overline{[(I-zT^*)^{-1}g_p, g_q]}=\overline{[f_p(z), f_q]},\end{aligned}$$

and

$$[f_{n+p}(\bar{z}), f_q] = [(I-\bar{z}T^*)^{-1} g_p, Kg_q] =$$
$$= -[K\cdot K(I-\bar{z}T^*)^{-1} g_p, Kg_q] = -\overline{[(I-zT^*)^{-1} Kg_p, g_q]} =$$
$$= -\overline{[f_{n+p}(z), f_q]}.$$

Thus $G_T(z)\in Q_{2n}(T)$, and in a similar way we get that $G_T'(z)\in Q_{2n}(T)$. By virtue of Lemma 2 we find that $\Omega_T(z)\in Q_{2n}(T)$.

Conversely, suppose $\Omega_T(z)\in Q_{2n}(T)$ for a $T\in O^s(V)$.

Then by virtue of the previous part of the theorem, we find that $\Omega(z)\in Q_{2n}(T^s)$.

From the relation (3.4) which could be written in the form

$$\Omega_T(z)+\Omega(z)\,\tau^*\,\Omega_T(z) = \tau+\Omega(z), \tag{3.5}$$

then we can derive that the constant matrix $\tau\in M_{2n}(C)$ belongs to the class $P_{2n}(C)$.

Hence we have $[TKg_p, Kg_q'] = \overline{[Tg_p, g_q']}$, $[TKg_p, g_q'] = -\overline{[Tg_p, Kg_q']}$ $(p, q = 1, \dots, n)$, wherefrom we conclude that $[(K^{-1}TK-T)g_p, f_q'] = 0$ $(p=1, \dots, n;\ q=1, \dots, 2n)$, thus $(K^{-1}TK-T)g_p = 0$, because this last vector belongs to $R(V)^{\perp}$. It is then immediately $(K^{-1}TK-T)f_p = 0$ for every $p=1, \dots, 2n$, thus $K^{-1}TKx = Tx\ (x\in D(V)^{\perp})$. But then $K^{-1}TKx = Tx$ for every $x\in H^s$, consequently $T\in B(H)$, thus $T\in O(V)$. □

4.

If U is a unitary extension from $O(V)$ which maps f_p into f_p' $(p=1, \dots, 2n)$ (obviously a unitary operator from $O(V)$), and $E^s(t)$ $(O\leqslant t\leqslant 2\pi)$ the spectral family of U in H^s, we have the family of hermitian matrices $\zeta(t) = \{\zeta_{p,q}(t)\}\in M_{2n}$ defined by $\zeta_{p,q}(t) = [E^s(t)f_p, f_q]$ $(1\leqslant p, q\leqslant 2n)$, and then

$$\int_0^{2\pi} d\zeta_{p,q}(t) = \delta_{p,q} \qquad (1\leqslant p, q\leqslant 2n)$$

([2], p. 253).

Theorem 2. *The matrix function* $\zeta(t)$ $(0\leqslant t\leqslant 2\pi)$ *has the following properties*:

$$\text{(a)}\qquad \overline{\zeta_{n+p,n+q}(t)} = \delta_{p,q} - \zeta_{p,q}(2\pi - t);$$

$$\text{(b)}\qquad \overline{\zeta_{n+p,q}(t)} = \zeta_{p,n+q}(2\pi - t) \qquad (1\leqslant p, q\leqslant n).$$

Proof. We have first that for every $1\leqslant p, q\leqslant n$:

$$\overline{\zeta_{n+p,n+q}(t)} = \overline{[E^s(t)Kg_p, Kg_q]} = \overline{[K(K^{-1}E^s(t)K)g_p, Kg_q]} =$$
$$= [(K^{-1}E^s(t)K)g_p, g_q].$$

We shall now apply a result of K. Viswanath [5] related to the spectral family of normal operators.

Let $F=F^s$ be the spectral measure of the unitary operator U (in H^s) and $C[0, t]$ $(0<t<2\pi)$ the segment of arc of the unit circle $(0\leqslant\lambda\leqslant t)$; then $E^s(t)=\int_{C[0,t]} dF(\Delta)$, and $E^s(0)=0$, $E^s(2\pi)=I$.

By virtue of a result of Viswanath ([5], p. 342) we have that $K^{-1}F(\Delta)K=F(\bar\Delta)$ for every Borel set Δ in the complex plane, so that

$$K^{-1}E^s(t)K=\int_{C[0,t]} dK^{-1}F(\Delta)K=\int_{C[0,t]} dF(\bar\Delta)=$$

$$=\int_{C[2\pi-t,\,2\pi]} dF(\Delta).$$

Now, by covering the unit circle with a family of halfopen rectangles (included down verteces), we get

$$\int_{C[0,l]} dF(\Delta)=\int_{C[l,\,2\pi]} dF(\Delta)+\int_{C[0,\,2\pi]} dF(\Delta)=E^s(2\pi)=I \qquad (0<l<2\pi),$$

so that

$$K^{-1}E^s(t)K=I-\int_{C[0,\,2\pi-t]} dF(\Delta)=I-E^s(2\pi-t). \tag{4.1}$$

Since this relation holds for $t=0$ and $t=2\pi$ too, it is satisfied for every $t\in[0, 2\pi]$.

Consequently we get

$$\overline{\zeta_{n+p,n+q}(t)}=[(I-E^s(2\pi-t))g_p, g_q]=$$

$$=\delta_{p,q}-\zeta_{p,q}(2\pi-t) \qquad (1\leqslant p, q\leqslant n).$$

In a similar way we find for every $1\leqslant p, q\leqslant n$:

$$\overline{\zeta_{n+p,q}(t)}=\overline{[E^s(t)Kg_p, g_q]}=\overline{[K(K^{-1}E^s(t)K)g_p, KK^{-1}g_q]}=$$

$$=[(I-E^s(2\pi-t))g_p, K^{-1}g_q]=$$

$$=-[g_p, Kg_q]+[E^s(2\pi-t)g_p, Kg_q]=$$

$$=\zeta_{p,n+q}(2\pi-t). \quad \square$$

Let next Q_{2n} be the set of all matrix functions $S(z)\in M_{2n}(C)$ of the form (*) whose domain is the open disc $|z|<1$.

Theorem 3. *The matrix function*

$$\mathcal{U}(z)=\int_0^{2\pi}(e^{it}+z)(e^{it}-z)^{-1}\,d\zeta(t)$$

belongs to the class Q_{2n}.

Proof. We have for every $1\leqslant p,\ q\leqslant n$,

$$\overline{\mathcal{U}_{n+p,n+q}(\bar z)}=\int_0^{2\pi}(e^{-it}+z)(e^{-it}-z)^{-1}\,d\,\overline{\zeta_{n+p,n+q}(t)}$$

$$\overline{\mathcal{U}_{n+p,q}(\bar z)}=\int_0^{2\pi}(e^{-it}+z)(e^{-it}-z)^{-1}\,d\,\overline{\zeta_{n+p,q}(t)},$$

wherefrom, using properties (a), (b) of the matrix $\zeta(t)$, and substituting t by $2\pi-t\,(0\leqslant t\leqslant 2\pi)$, it follows

$$\overline{\mathcal{U}_{n+p,n+q}(\bar z)}=\mathcal{U}_{p,q}(z),$$

$$\overline{\mathcal{U}_{n+p,q}(\bar z)}=-\mathcal{U}_{p,n+q}(z)\qquad(|z|<1),$$

thus $\mathcal{U}(z)=\{\mathcal{U}_{p,q}(z)\}\in Q_{2n}$. □

5.

Lemma 3. *The complex Hilbert space H^s possesses the structure of a left Wachs space if and only if there exists in H^s an antilinear involutive operator $K\in B(H^s)$ with the property*

$$[Kx,\,Ky]=\overline{[x,\,y]}\qquad(x,\,y\in H^s).$$

Proof. This condition is obviously necessary because in left Wachs spaces H the operator $Kx=jx\,(x\in H)$ has all required properties.

Conversely let us suppose there exists such an operator on a left complex Hilbert space H. If we introduce in H^s the quaternionic structure by $qx=(z_1+jz_2)\,x=z_1x+\bar z_2Kx\,(x\in H)$, and quaternionic bilinear form by $\langle x,y\rangle=[x,y]-j[Kx,y]\,(x,y\in H)$, it is easily verified that H^s becomes a Wachs space H, and the quaternionic scalar product is according to the norm $\|x\|=\sqrt{[x,x]}$ (because $[x,Kx]\equiv 0$). □

Lemma 4. *Let H^s_{2n} be the left vector space of all $2n$-tuples $x=(\xi_1,\dots,\xi_{2n})$ of complex numbers, with the scalar product*

$$[x,\,y]=\sum_{p=1}^{2n}\xi_p\,\bar\eta_p.$$

Then defining the quaternionic structure by

$$Kx = K(\xi_1, \ldots, \xi_{2n}) = (-\overline{\xi}_{n+1}, \ldots, \overline{\xi}_{2n}, \overline{\xi}_1, \ldots, \overline{\xi}_n)$$

the space H_{2n}^s becomes a left Wachs space.

Proof. We omit proving that K has the properties required in the previous lemma. $\square$

We next mention that if $\hat{e}_p = (\delta_{p,1}, \ldots, \delta_{p,2n})$ $(p=1, \ldots, 2n)$, then $\hat{e}_{n+p} = K\hat{e}_p$ $(p=1, \ldots, n)$, and the set $\{\hat{e}_1, \ldots, \hat{e}_n\}$ is an o.n.b. of the corresponding Wachs space H_{2n}^s. On the other hand, it is isomorphic to the left Wachs space $H_n(Q)$.

Statement 1. *The matrices $\Omega(z)$, $\mathcal{U}(z)$ $(|z|<1)$, as the operators in H_{2n}^s, have for every $|z|<1$ the following properties:*

(a) $\|\Omega(z)\|<1$,

(b) $\operatorname{Re}\mathcal{U}(z)>0$.

In general case, the operators $\Omega(z)$, $\mathcal{U}(z)$ are not linear on H_{2n} but they are linear at least for real values of z.

Proof. The proof of (a), (b) given in [2], we only verify the last statement.

By definition, we have that

$$[\Omega(z)\hat{e}_p, \hat{e}_q] = \Omega_{p,q}(z) \qquad (|z|<1;\ p, q = 1, \ldots, 2n),$$

thus

$$\Omega(z)\hat{e}_p = \sum_{q=1}^{2n} \Omega_{p,q}(z)\hat{e}_q \qquad (p=1, \ldots, 2n).$$

We next find for $1 \leqslant p \leqslant n$,

$$\begin{aligned} K^{-1}\Omega(z)K\hat{e}_p &= K^{-1}\Omega(z)\hat{e}_{p+n} = \\ &= \sum_{q=1}^{n} K^{-1}\Omega_{n+p,q}(z)\hat{e}_q + \sum_{q=1}^{n} K^{-1}\Omega_{n+p,n+q}(z)K\hat{e}_q = \\ &= \sum_{q=1}^{n} \overline{\Omega_{n+p,n+q}(z)}\,\hat{e}_q + \sum_{q=1}^{n} \overline{(-\Omega_{n+p,q}(z))}\,K\hat{e}_q = \\ &= \sum_{q=1}^{n} \Omega_{p,q}(\bar{z})\hat{e}_q + \sum_{q=1}^{n} \Omega_{p,n+q}(\bar{z})K\hat{e}_q = \Omega(\bar{z})\hat{e}_p. \end{aligned}$$

Since the like holds for $p=n+1, \ldots, 2n$, we see that $K^{-1}\Omega(z)K = \Omega(\bar{z})$. Thus for $z \in R$, $\Omega(z)$ commutes with K and consequently $\Omega(z) \in B(H_{2n})$. $\square$

We mention the following theorem without proof too.

Theorem 4. *Two simple isometric operators V and V_1 with deficiency indices (n, n) are unitarly equivalent if and only if their c.m.f. are connected by the following relation*

$$\Omega(z) = u\,\Omega_1(z)\,v \qquad (|z|<1),$$

where u, v are constant unitary matrices of the class P_{2n}. □

6.

Let next L^s be the vector space of all complex continous functions $\hat{f}: [0, 2\pi] \to H^s_{2n}$, $\hat{f}(t) = (f_1(t), \dots, f_{2n}(t))$ defined on segment $[0, 2\pi]$, and for a hermitian matrix function $\sigma(t) = \{\sigma_{p,q}(t)\}$ of order $2n$, let us define ([2])

$$(**) \qquad [\hat{f}, \hat{g}] = \sum_{p,q=1}^{2n} \int_0^{2\pi} f_p(t)\overline{g_q(t)}\, d\sigma_{p,q}(t).$$

Together with conditions considered in [2] i.e. $\sigma_{\hat{z}}(t) = \sum_{p,q=1}^{2n} \sigma_{p,q}(t)\cdot z_p \bar{z}_q$ is right continuous and non-decreasing in t $(0 \leqslant t \leqslant 2\pi,\ \hat{z} = (z_1, \dots, z_{2n}) \in C^{2n})$ and $\int_0^{2\pi} d\sigma_{p,q}(t) = \delta_{p,q}$ $(1 \leqslant p,\ q \leqslant 2n)$, we additionally suppose that the matrix function $\sigma(t)$ satisfies for every $t \in [0, 2\pi]$ following conditions:

$$(6.1) \qquad \begin{cases} \overline{\sigma_{n+p,n+q}(t)} = \delta_{p,q} - \sigma_{p,q}(2\pi - t), \\ \overline{\sigma_{n+p,q}(t)} = \sigma_{p,n+q}(2\pi - t) \end{cases}$$

$(1 \leqslant p,\ q \leqslant n)$.

Statement 2. *The space L^s is a (non complet) Hilbert space with scalar product* $(**)$.

If H^s is the corresponding completion of L^s, introduce the quaternionic structure in L^s by

$$(K\hat{f})_p(t) = \begin{cases} -f_{n+p}(2\pi - t), & 1 \leqslant p \leqslant n \\ f_{p-n}(2\pi - t), & n+1 \leqslant p \leqslant 2n \end{cases},$$

and in H^s by continuity.

Then H^s becomes a left Wachs space.

Proof. We prove only the last assertion, because all that remained is due to Livšits [2].

Since almost all conditions from Lemma 3 are easily verified, we only verify that relation $[Kf, Kg]=\overline{[f, g]}$ on the space H^s, or (which is sufficient) on the subspace L^s, holds.

$$\text{If } \hat{f}(t)=(f_1(t), \ldots, f_{2n}(t)), \quad \hat{g}(t)=(g_1(t), \ldots, g_{2n}(t)),$$

then

$$[\hat{f}, \hat{g}]=\sum_{p,q=1}^{n} \int_0^{2\pi} \{f_p(t)\overline{g_q(t)}\,d\sigma_{p,q}(t)+f_p(t)\overline{g_{n+q}(t)}\,d\sigma_{p,n+q}(t)+$$

$$+f_{n+p}(t)\overline{g_q(t)}\,d\sigma_{n+p,q}(t)+f_{n+p}(t)\overline{g_{n+q}(t)}\,d\sigma_{n+p,n+q}(t)\},$$

$$[K\hat{f}, K\hat{g}]=\sum_{p,q=1}^{n} \int_0^{2\pi} \{\overline{f_{n+p}(2\pi-t)}\,g_{n+q}(2\pi-t)\,d\sigma_{p,q}(t)-$$

$$-\overline{f_{n+p}(2\pi-t)}\,g_q(2\pi-t)\,d\sigma_{p,n+q}(t)-$$

$$-\overline{f_p(2\pi-t)}\,g_{n+q}(2\pi-t)\,d\sigma_{n+p,q}(t)+$$

$$+\overline{f_p(2\pi-t)}\,g_q(2\pi-t)\,d\sigma_{n+p,n+q}(t)\}.$$

Using the properties (6.1) of the matrix function $\sigma(t)$, and applying them to each of addends in above expressions, we directly get the required property. □

Denoting further

$$L(i)=\{\hat{f}\in L^s: f_{n+p}\equiv 0, p=1, \ldots, n\},$$

$$H(i)=\text{completion of } L(i) \text{ (in } H^s)$$

(the smallest Hilbert subspace of H^s containing $L(i)$), we get

$$KL(i)=\{\hat{f}\in L^s: f_p\equiv 0,\ p=1, \ldots, n\}, \quad \text{and}$$

$$L^s=L(i)\dot{+}KL(i), \quad H^s=H(i)\dot{+}KH(i).$$

Remark. In general case, $H(i)$ and $KH(i)$ are not orthogonal (in H^s), and the orthogonality occurs if and only if

$$\sigma_{p,n+q}(t)\equiv\sigma_{n+p,q}(t)\equiv 0 \qquad (0\leqslant t\leqslant 2\pi;\ 1\leqslant p, q\leqslant n),$$

thus $\sigma(t)$ is of the form

$$\sigma(t)=\begin{bmatrix}\sigma_1(t) & 0\\ 0 & \sigma_4(t)\end{bmatrix}. \quad \square$$

Further let H be a left Hilbert space, $H^s=H(i)\dot{+}KH(i)$, when $H(i)$ is a complex Hilbert subspace of H, $U\in B(H)$ be a unitary operator on H, an extension of a complex unitary operator $U(i)\in B(H(i))$.

Lemma 5. *If $E_i^s(t)$ and $E^s(t)$ $(0 \leqslant t \leqslant 2\pi)$ are the spectral families of operators $U(i)$, U in spaces $H(i)$, H^s respectively, then $E_i^s(t) = E^s(t)|_{H(i)}$, and for $x = x_0 + Kx_1$ $(x_0, x_1 \in H(i))$ the next holds*

$$E^s(t)\,x = E_i^s(t)\,x_0 + K(I - E_i^s(2\pi - t))\,x_1. \tag{6.2}$$

Proof. Let us put $D^s(t) = E^s(t)|_{H(i)}$. Then $D^s(t)$ $(0 \leqslant t \leqslant 2\pi)$ is a family of symmetric operators on $H(i)$, increasing in $t \in [0, 2\pi]$, $D^s(0) = 0$, $D^s(2\pi) = I_{H(i)}$, and from

$$[Ux, y] = \int_0^{2\pi} e^{it}\, d[E^s(t)\,x, y] \qquad (x, y \in H^s)$$

we get especially

$$[Ux_0, y] = \int_0^{2\pi} e^{it}\, d[E^s(t)\,x_0, y] \qquad (x_0 \in H(i),\ y \in H). \tag{6.3}$$

Since

$$[U(i)\,x_0, y_0] = \int_0^{2\pi} e^{it}\, d[E_i^s(t)\,x_0, y_0] \qquad (x_0, y_0 \in H(i)), \tag{6.4}$$

the uniqeness of the spectral family of $U(i)$ implies that $E_i^s(t) = D^s(t)\,(0 \leqslant t \leqslant 2\pi)$.

Using this, from (6.3) we get the following (important) relation:

$$[U(i)\,x_0, y] = \int_0^{2\pi} e^{it}\, d[E_i^s(t)\,x_0, y] \qquad (x_0 \in H(i),\ y \in H^s). \tag{6.3'}$$

Further, if $x = Kx_0 \in KH(i)$, we obtain

$$E^s(t)\,x = E^s(t)\,(Kx_0) = K(K^{-1}E^s(t)\,K)\,x_0 = K(I - E^s(2\pi - t))\,x_0 =$$
$$= K(I - E_i^s(2\pi - t))\,x_0,$$

which completes the proof. □

Remark. Here we do not assume that $H(i)$ reduces U (in H^s), that is $H(i) \perp_s KH(i)$. □

Theorem 5. *A matrix function $\Omega(z)$ of order $2n$ defined for $|z| < 1$, is a Livšits characteristic matrix function of an isometric operator V with deficiency indices (n, n), if and only if it satisfies:*

(1°) $\Omega(0) = 0$;

(2°) $\Omega(z)$ *is analytic for* $|z| < 1$;

(3°) $\|\Omega(z)\| < 1$ *for every* $|z| < 1$ *(in the space H_{2n}^s)*;

(4°) $\Omega(z) \in Q_{2n}$.

Proof. These relations are obviously necessary.

Next let us suppose that $\Omega(z)$ satisfies these conditions, and

$$\mathcal{U}(z) := [I - \Omega(z)]\,[I + \Omega(z)]^{-1} \qquad (|z| < 1)$$

(thus $\mathcal{U}(z) \in B(H^s_{2n})$).

By virtue of the Lemma 2, the matrix $\mathcal{U}(z) \in Q_{2n}$.

As in [2], we find a hermitian matrix function $\sigma(t) = \{\sigma_{p,q}(t)\}$ $(0 \leqslant t \leqslant 2\pi)$ with the property

$$(6.5) \qquad [\mathcal{U}(z)\hat{f}, \hat{g}] = \int_0^{2\pi} (e^{it} + z)(e^{it} - z)^{-1}\, d[\sigma(t)\hat{f}, \hat{g}] \qquad (\hat{f}, \hat{g} \in H^s_{2n}),$$

thus

$$\mathcal{U}(z) = \int_0^{2\pi} (e^{it} + z)(e^{it} - z)^{-1}\, d\sigma(t),$$

and $\int_0^{2\pi} d\sigma(t) = I(H_{2n})$.

We assert that the matrix $\sigma(t)$ possesses the following properties:

$$(a') \qquad \overline{\sigma_{n+p,n+q}(t)} = \delta_{p,q} - \sigma_{p,q}(2\pi - t);$$

$$(b') \qquad \overline{\sigma_{n+p,q}(t)} = \sigma_{p,n+q}(2\pi - t) \qquad (0 \leqslant t \leqslant 2\pi).$$

Since

$$\overline{\mathcal{U}_{n+p,n+q}(\bar{z})} = \int_0^{2\pi} (e^{-it} + z)(e^{-it} - z)^{-1}\, d\,\overline{\sigma_{n+p,n+q}(t)} =$$

$$= \int_0^{2\pi} (e^{-it} + z)(e^{-it} - z)^{-1}\, d(\delta_{p,q} - \overline{\sigma_{n+p,n+q}(2\pi - t)}),$$

$$\mathcal{U}_{p,q}(z) = \int_0^{2\pi} (e^{it} + z)(e^{it} - z)^{-1}\, d\sigma_{p,q}(t),$$

from the fact that $\mathcal{U}(z) \in Q_{2n}$ and uniquiness of the matrix $\sigma(t)$, we obtain

$$\sigma_{p,q}(t) = \delta_{p,q} - \overline{\sigma_{n+p,n+q}(2\pi - t)} \qquad (0 \leqslant t \leqslant 2\pi).$$

In a similar way, we prove the property (b').

Next let L^s, $L(i)$, H^s and H^s be the spaces from Statement 2.

Define an operator U on L^s by

$$(U\hat{f})(t) = \begin{cases} e^{it}\hat{f}(t), & \hat{f} \in L(i) \\ e^{-it}\hat{f}(t), & \hat{f} \in KL(i) \end{cases},$$

and further on H^s (i.e. on H) by continuity.

The operator U is a unitary operator on H.

Let $E^s(t)$ $(0\leqslant t\leqslant 2\pi)$ be the spectral family of U on H^s, and hence $E_i^s(t)=E^s(t)\,|_{H(i)}$ the spectral family of unitary operator $U(i)=U\,|_{H(i)}$ in $H(i)$.

Next let $\hat{e}_p=(\delta_{p,1},\ldots,\delta_{p,2n})$ where $\delta_{p,q}(t)\equiv\delta_{p,q}(p,q=1,\ldots,2n)$. Then $\hat{e}_p\in L(i)$ $(p=1,\ldots,n)$, and $\hat{e}_{n+p}=K\hat{e}_p\,(p=1,\ldots,n)$.

By virtue of the relation (6.3′), entirely equal to [2] (p. 257), we obtain that for any interval $\Delta\subseteq[0,2\pi]$,

$$[E^s(\Delta)\,\hat{e}_p,\hat{e}_q]=\int_\Delta d[E_i^s(t)\,\hat{e}_p,\hat{e}_q]=\int_\Delta d[E^s(t)\,\hat{e}_p,\hat{e}_q]=$$

$$=\int_\Delta d\sigma_{p,q}(t)$$

for $1\leqslant p\leqslant n$; $1\leqslant q\leqslant 2n$.

Therefrom: $\sigma_{p,q}(t)=[E^s(t)\,\hat{e}_p,\hat{e}_q]$ $(1\leqslant p\leqslant n;\ 1\leqslant q\leqslant 2n)$.

It remains to prove that the above equality holds for $n+1\leqslant p\leqslant 2n$, $1\leqslant q\leqslant 2n$ also.

But properties (a′), (b′) of the matrix $\sigma(t)$, and relations (a), (b) from Theorem 2 imply that

$$\sigma_{p,q}(t)=[E^s(t)\,\hat{e}_p,\hat{e}_q]$$

for every $1\leqslant p,q\leqslant 2n$.

Consequently,

$$\mathcal{U}_{p,q}(z)=\int_0^{2\pi}(e^{it}+z)(e^{it}-z)^{-1}\,d[E^s(t)\,\hat{e}_p,\hat{e}_q]\qquad(1\leqslant p,q\leqslant 2n). \tag{6.6}$$

If $G=\{\hat{f}\in H\colon\langle e_p,f\rangle=0,\ p=1,\ldots,n\}$, that is

$$G=\{\hat{f}\in H\colon[\hat{e}_p,\hat{f}]=0,\ p=1,\ldots,2n\},$$

and $V\hat{f}=U\hat{f}$ $(\hat{f}\in G)$, then by virtue of (6.6), $\Omega(z)$ is a Livšits c.m.f. of the operator V. $\square$

7.

Among the other “characteristic” properties of c.m.f. operators on Wachs spaces, we shall mention only the following property.

Let us introduce the functions

$$k_1(z)=\det(\Omega(z)+\tau)\qquad(|z|\leqslant 1),$$

$$k_2(z)=\det(I_{2n}+\Omega(1/\bar{z})\,\tau^*)\qquad(|z|\geqslant 1),$$

where τ is the “matrix” of an arbitrary operator $T\in O(V)$.

Since the spectrum of an arbitrary operator $T \in B(H)$ is an r-symmetric set, in view of the Theorem 6 in [2], we may expect that

$$k_\nu(z)=0 \quad \text{iff} \quad k_\nu(\bar{z})=0 \qquad (\nu=1, 2).$$

Really, we have the next assertion.

Theorem 6. *For $|z| \leqslant 1$, and $|z| \geqslant 1$ respectively, the following equalities hold*:

$$k_\nu(\bar{z})=\overline{k_\nu(z)} \qquad (\nu=1, 2). \tag{6.7}$$

Thus $k_\nu(z)$ is real for every real z.

Proof. Prior to proving this, we shall prove the following property of block-matrices: If the matrices $A, B, C, D \in M_n(C)$, then

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} = \begin{vmatrix} A & -C \\ -B & D \end{vmatrix} = \begin{vmatrix} D & -B \\ -C & A \end{vmatrix}.$$

Indeed, when substituting the first and the second columns, and then the first and the second rows in the third block-matrix, we get

$$\begin{vmatrix} D & -B \\ -C & A \end{vmatrix} = (-1)^n \begin{vmatrix} -B & D \\ A & -C \end{vmatrix} = (-1)^{2n} \begin{vmatrix} A & -C \\ -B & D \end{vmatrix} = \begin{vmatrix} A & -C \\ -B & D \end{vmatrix}.$$

Denoting next by a, b, c, d respectively the numbers of elements of the matrix $\begin{bmatrix} A & -C \\ -B & D \end{bmatrix}$ which are elements of A, $-B$, $-C$, D respectively, in the general term of the development of determinant $\begin{vmatrix} A & -C \\ -B & D \end{vmatrix}$, it is not hard to see that $a+b=c+d=a+c=b+d$ holds, so that $a=d$, $b=c$.

Hence, in this term, in comparison with the corresponding term of the determinant $\begin{vmatrix} A & C \\ B & D \end{vmatrix}$ the factor $(-1)^c(-1)^b=(-1)^{2b}=+1$ appears, so that they coincide.

Consequently,

$$\begin{vmatrix} A & -C \\ -B & D \end{vmatrix} = \begin{vmatrix} A & C \\ B & D \end{vmatrix},$$

which completes the proof.

Next, denote by Φ: $M_{2n}(C) \to M_{2n}(C)$ the transformation which maps the block matrix $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$ into $\begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$.

By virtue of the Theorem 1, we see that the c.m.f. of an arbitrary operator $T \in O(V)$ satisfies the relation $\Omega(\bar{z})=\overline{\Phi\Omega(z)}$ ($|z| \leqslant 1$), and $\tau=\overline{\Phi(\tau)}$ holds.

Since the mapping Φ is a homomorphism, we easily find that

$$k_1(\bar{z}) = \det(\Omega(\bar{z}) + \tau) = \det(\overline{\Phi\Omega(z)} + \overline{\Phi(\tau)}) =$$

$$= \overline{\det(\Phi(\Omega(z) + \tau))} = \overline{\det(\Omega)(z) + \tau},$$

thus $k_1(\bar{z}) = \overline{k_1(z)}$ for $|z| \leqslant 1$.

Similarly,

$$k_2(\bar{z}) = \det(I_{2n} + \overline{\Phi\Omega(1/\bar{z})\Phi(\tau^*)}) =$$

$$= \overline{\det(\Phi(I + \Omega(1/\bar{z})\tau^*))} = \overline{\det(I + \Omega(1/\bar{z})\tau^*)} =$$

$$= \overline{k_2(z)}, \qquad (|z| \geqslant 1)$$

q.e.d. □

Note. We wish to announce further investigations of the characteristic matrix functions of general linear operators, or contractions, relevant and inspired by the papers of V. M. Brodskiĭ, M. S. Livšits, A. V. Štrauss, Yu. L. Šmuljan, B. Sz. Nagy, C. Foias, etc.

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Math. Institute
Faculty of Sciences,
Belgrade, Studenski trg 16.