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LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS IN A FIELD I

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In this paper we shall study the differential equation

$$\sum_{\mu=0}^{M} A_\mu x^{(\mu)}(\lambda)=0, \quad \lambda_1 \leqslant \lambda \leqslant \lambda_2, \tag{1}$$

where the coefficients A_μ are of the form:

$$A_\mu = \sum_{k=0}^{\infty} \mathbf{a}_{\mu,k}\, e^{-\tau_k^\mu s} \tag{2}$$

For every $\mu=0, \ldots, M$ $\{\tau_k^\mu\}$ is a strict monotone increasing sequence which diverges, $\tau_0^{(\mu)} > -\infty$, $\mathbf{a}_{\mu,0} \neq 0$ and

$$\mathbf{a}_{\mu,k} = \sum_{\nu=\nu_{\mu,k}}^{\infty} \alpha_{\mu,k,\nu}\, l^{\frac{\nu}{\sigma_{\mu,k}}}, \quad \alpha_{\mu,k,\nu_{\mu,k}} \neq 0, \tag{3}$$

$\nu_{\mu,k} > -\infty$, $\sigma_{\mu,k} \in N$. Here **s** is the differential operator l the integral operator and $e^{-\lambda s}$, $\lambda > 0$ the translation operator in the field $\mathcal{M}$ of Mikusiński operators [1].

In the first part we shall give the conditions of the existence of the linearly independent solutions of equation (1) and we shall compare our results with the results obtained by J. Wloka [7]. We shall show that the existence and the number of the solutions can be read from a broken line which is easy for constructions. Then we shall restrict a little our equation (1) and we shall give for it the effective construction of the linearly independent solutions and the character of the solutions (are they elements from $\mathcal{C}$, $\mathcal{L}$, $\mathcal{D}'$, or only $\mathcal{M}$).

In the second part we shall give the approximations of the solutions and the error's estimation, and the application to the partial differential-difference equations.

All these results are generalisations of some earlier [2] [3].

We use all the definitions as in the book of Mikusiński [1].

1. Notations. Let $\mathcal{L}$ be the ring of local integrable functions over $[0, \infty)$, with the operations: sum and finite convolution. We denote by $\mathbf{f}$ or $\{f(t)\}$ the element from $\mathcal{M}$ which corresponds to $f(t) \in \mathcal{L}$. The relation $\mathbf{f} \geqslant_T \mathbf{g}$ or $\mathbf{f} =_T \mathbf{g}$ means $f(t) \geqslant g(t)$ or $f(t) = g(t)$ for $t \in [0, T]$, $T < \infty$. $\mathcal{C}$ is the ring of continuous functions over $[0, \infty)$. $\mathbf{H}_\lambda$ is the Heaviside function:

$$\mathbf{H}_\lambda = H(t-\lambda) = \begin{cases} 0, & 0 \leqslant t < \lambda \\ 1, & t \geqslant \lambda \geqslant 0 \end{cases}$$

$\mathbf{s}\mathbf{H}_\lambda = e^{-\lambda s} \cdot F(\beta) = \{F(\beta, t)\}$ is the function $\{t^{-\beta-1}\Phi(-\beta, -\sigma, -t^{-\sigma})\}$, $t > 0$, $0 < \sigma < 1$, where Φ is the function of E. M. Wright [5]. The property of this function we shall use is the following $\mathbf{s}^\gamma \mathbf{F}(0) = \mathbf{F}(\gamma)$, $\gamma \in R$.

Let $P(l)$ be a polynomial in the integral operator l with the coefficients complex numbers. It is well known [1] that the equation

$$\sum_{i=0}^{n} P_i(l)\, \mathbf{a}^i = 0$$

has n solutions in $\mathcal{M}$ and these solutions are of the form

$$\mathbf{a} = \sum_{i=i_0}^{n} \alpha_i\, l^{i/s}, \quad \alpha_0 \neq 0, \quad i_0 > -\infty, \quad \alpha_i \text{ complex numbers.} \tag{4}$$

This series converges in $\mathcal{M}$. Every such solution (4) is named algebraic operator. We know that $\mathbf{a} = 0 \Leftrightarrow \alpha_i = 0$ $i = i_0, i_0 + 1, \ldots$ Let $\mathcal{A}$ be the set of all algebraic operators. $\mathcal{A}$ is a field which is algebraically closed. $\mathcal{A}_0$ is a subfield of those elements for which $i_0 = 0$ and $\alpha_0 \neq 0$. In $\mathcal{A}$ we have the following proposition:

Proposition A. The element $\mathbf{a} \in \mathcal{A}$ is a logarithm if and only if $i_0/s > -1$ or $i_0/s = -1$ and α_0 a real number.

2. **The field** $\mathcal{H}$

Let $\{\tau_i\}$ be a sequence of real numbers which is strict monotone increasing and diverges; $\tau_0 > -\infty$ and $\{\mathbf{a}_i\}$ is a sequence from $\mathcal{A}$, $\mathbf{a}_0 \neq 0$. The set of the series of the form:

$$\sum_{i=0}^{\infty} \mathbf{a}_i\, e^{-\tau_i s}$$

with the usual sum and product is the field $\mathcal{H}$. For the elements of the field $\mathcal{H}$ we know

Proposition B.

1. The series $\sum_{i=0}^{\infty} \mathbf{a}_i\, e^{-\tau_i s}$ converges in $\mathcal{M}$,

2. $\sum_{i=0}^{\infty} \mathbf{a}_i\, e^{-\tau_i s} = 0 \Leftrightarrow \mathbf{a}_i = 0,\ i = 0, 1, \ldots$

3. The operator given by the series $\sum_{i=0}^{\infty} \mathbf{a}_i\, e^{-\tau_i s}$ is a logarithm if and only if $\tau_0 > 0$ or $\tau_0 = 0$ and $\mathbf{a}_0$ is a logarithm.

Proposition C. $\mathcal{H}$ is an algebraically closed field.

We shall introduce the notation $\mathcal{H}_0$ for a subfield of $\mathcal{H}$ for which elements $\tau_0 = 0$ and $a_0 \neq 0$, and $\mathcal{H}'$ for subring of $\mathcal{H}$ when $\tau_0 \geqslant 0$.

Let α be a fix positive number, and $\{\alpha_i\}$ a sequence of complex numbers. Let $\mathbf{D}$ be a series

$$\mathbf{D} = \sum_{i=i_0}^{\infty} \alpha_i\, e^{-i\alpha s} = \sum_{i=i_0}^{\infty} \alpha_i\, \mathbf{h}^i, \quad \mathbf{h} = e^{-\alpha s} \tag{5}$$

where i_0 is an integer or zero. The series of the form (5) are the elements of a subfield $\mathcal{H}(\alpha)$ of $\mathcal{H}$. The notations $\mathcal{H}_0(\alpha)$ and $\mathcal{H}'(\alpha)$ have the correspendent meaning to $\mathcal{H}_0$ and $\mathcal{H}'$.

3. The infinum curve for a set of points in $\mathcal{R}^2$

For a point $(x_0, y_0) \in \mathcal{R}^2$ we shall say that it is not under the curve $y = f(x)$ if $y_0 \geqslant f(x_0)$. For a broken line S we shall say that it is an infinum curve for a set of points $T \subset \mathcal{R}^2$ if in every its apex is one of the points of T and if all the points of T are not under S. Let us note this broken line by $S(m^{(1)}, \ldots, m^{(k)})$ to stress that it is composed of segments with the coefficients of directions $m^{(1)}, \ldots, m^{(k)}$, respectivelly.

Definition 1. For the set of points $T \equiv \{(x_i, y_i)\}_{i \in I}$ we shall say that the number m is a solution of the system (6) if there exist at least two values for $i : i'$ and i'' such that

$$y_{i'} + m\, x_{i'} = y_{i''} + m\, x_{i''} \leqslant y_i + m\, x_i, \quad i \in I. \tag{6}$$

In geometrical language this definition says that for the set of points $T \equiv \{(x_i, y)\}_{i \in I}$ the number m is a solution of the system (6) if there exist at least two points $(x_{i'}, y_{i'})$ and $(x_{i''}, y_{i''})$ from T laying on the straight line $y = -mx + b$ so that every parallel line $y = -mx + b'$ which contains any point $(x_i, y_i) \in T$ has its $b' \geqslant b$. That means that all the points $(x_i, y_i) \in T$ are not under the straight line $y = -mx + b$ $(y_i \geqslant -mx_i + b)$. It is now clear that all the numbers $-m$ which are solutions of the system (6) belong to the set of coefficients of directions $m^{(1)}, \ldots, m^{(k)}$ of the infinum curve $S(m^1, \ldots, m^{(k)})$ for the set T. All the points from T for which we have equality in (6) for a fixed m belong to the same segment of S.

In such a way to find the numbers m we can use a numerical method to solve equation (6), or the geometrical figure, as figure 1, can give us the asked value for m.

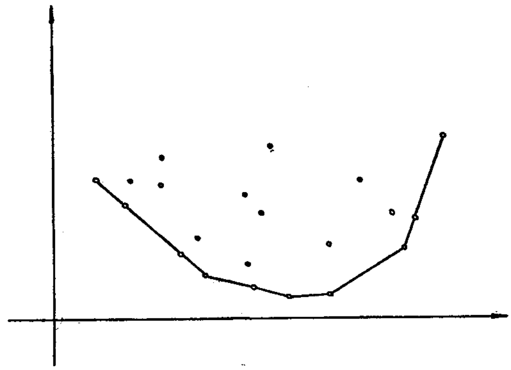

fig. 1

We shall use in the later the following evident lemma:

Lemma 1. *For a set $T \subset R^2$ which has a finite number of elements and more than one, system* (6) *has at least one solution.*

4. Existence of the linearly independent solutions of equation (1)

Let us consider differential equation (1). The equation

$$(7) \qquad \sum_{\mu=0}^{M} \mathbf{A}_\mu \mathbf{w}^\mu = 0, \quad \text{respectively} \quad \sum_{\mu=0}^{M} \sum_{k=0}^{\infty} \mathbf{a}_{\mu,k}\, e^{-\tau_k^\mu s}\, \mathbf{w}^\mu = 0$$

is the characteristic equation which corresponds to the equation (1). The proposition C says that equation (7) has M solutions in $\mathcal{H}$, all of the form

$$(8) \qquad \mathbf{w} = \sum_{i=0}^{\infty} \mathbf{c}_i\, e^{-\lambda_i s} = e^{-\lambda_0 s}\, \mathbf{v}, \quad \mathbf{v} \in \mathcal{H}_0, \quad \lambda_i \nearrow \infty$$

Proposition 1. Let $\{(\mu, \tau_0^\mu)\}_{\mu=0}^{M}$ be the set T for the characteristic equation (7) and $S(m^{(1)}, \ldots, m^{(r)})$ the infinum curve which corresponds to T.

1. If $m^{(i)} > 0$, $i = 1, 2, \ldots, r$, all the solutions of equation (7) are not logarithmic.

2. For every $m^{(i)} < 0$, equation (7) has at least one solution which is a logharithm and has a form of (8) where $\lambda_0 = -m^{(i)}$.

3. For $m^{(s)} = 0$ let (μ, τ_0^μ), $\mu = i, j, \ldots, t$, be the points from T which belong to te segment of S parallel to the x – axis and $\nu_{k,0}/\sigma_{k,0} \equiv \nu_k \sigma_k$, $k = i, j, \ldots, t$,

the numbers from relation (3). Let T' be the set of points $\left(k, \frac{\nu_k}{\sigma_k}\right)$, $k=i, j, \ldots, t$ and $S'(n^{(1)}, \ldots, n^{(p)})$ be the infinum curve for T', then w given by (8) is a logarithm for those and only those values of $n^{(i)}$ for which $n^{(i)} > -1$ or $n^{(i)} = -1$ and β_0 is real; β_0 is the first coefficient of $\mathbf{c}_0$.

Proof. — Let us suppose that a solution of the equation (7) is of the form (8), then we shall obtain

$$\sum_{\mu=0}^{M} \sum_{k=k_\mu}^{\infty} \mathbf{a}_{\mu,k}\, e^{-(\tau_k^\mu+\mu\lambda_0)s}\, \mathbf{v}^\mu = 0 \tag{9}$$

For a fixed $\mu = 0, 1, \ldots, M$ the least power of the translation operator in the equation (9) is $\tau_0^\mu + \lambda_0 \mu$ and the corresponding coefficient is $\mathbf{a}_{\mu,0} \neq 0$. Proposition **B** says that there exist at least two values for $\mu : i, j$ such that

$$\tau_0^i + \lambda_0\, i = \tau_0^j + \lambda_0\, j \leqslant \tau_0^\mu + \lambda_0\, \mu, \quad \mu = 0, \ldots, M \tag{10}$$

We know that this system has at least one solution for λ_0 (Lemma 1) and let us suppose that in (10) the equality is attained for $\mu = i, j, \ldots, t$, $i > j > \cdots > t$, then $\mathbf{c}_0$ can be obtained from

$$\mathbf{a}_{i,0}\, \mathbf{c}_0^i + \mathbf{a}_{j,0}\, \mathbf{c}_0^j + \cdots + \mathbf{a}_{t,0}\, \mathbf{c}_0^t = 0 \tag{11}$$

We know that $\mathcal{A}$ is algebraically closed field, so for $\mathbf{c}_0$ we have i-t solutions of the equation (11).

Points 1 and 2 of our proposition are the direct consequences of point 3 of proposition **B**. Point 3 of our proposition follows from the point 3 of proposition **B** and we have to determine: is a solution $\mathbf{c}_0$ of the equation (11) the logarithm or not.

Let $\mathbf{c}_0$ be of the form

$$\mathbf{c}_0 = \sum_{i=i_0}^{\infty} \beta_t\, l^{i/s} = l^{i_0/s}\, \mathbf{b}, \quad \mathbf{b} \in \mathcal{A}_0, \quad \beta_0 \neq 0$$

β_i are complex numbers and $a_{i,0} = l^{\nu_i/s_i} \mathbf{b}_i$, $\mathbf{b}_i \in \mathcal{A}_0$. Relation (11) can be written now

$$l^{\nu_i/s_i + (i_0/s)\, i}\, \mathbf{b}_i\, \mathbf{b}^i + l^{\nu_j/s_j + (i_0/s)\, j}\, \mathbf{b}_j\, \mathbf{b}^j + \cdots +$$
$$+ l^{\nu_t/s_t + (i_0/s)\, t}\, \mathbf{b}_t\, \mathbf{b}^t = 0.$$

The further procedure is the same as for the equation (10), that means we have to write the system which corresponds to the system (11). The end of the point 3 of our proposition follows from the proposition A.

We can compare our results with those of J. Wloka [7]. His main results is:

Let $m \geqslant m_k$ for $k=1, \ldots, d$. The equation

$$(12) \qquad \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda, t)}{\partial \lambda^{\mu} \partial t^{\nu}} + \sum_{k=1}^{d} \sum_{\mu=0}^{mk} \sum_{\nu=0}^{nk} \frac{\partial^{\mu+\nu} x(\lambda, t-\tau_k)}{\partial \lambda^{\mu} \partial t^{\nu}} = \varphi(\lambda, t)$$

$\lambda_1 \leqslant \lambda \leqslant \lambda_2$, $0 \leqslant t < \infty$, $\tau_k > 0$ for $k=1, \ldots, d$ and $x(\lambda, t)=0$ for $t<0$, is 1) logarithmic, 2) mixed or 3) pure if and only if the equation (the main part of the equation (12)):

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda, \tau)}{\partial \lambda^{\mu} \partial t^{\nu}} = 0$$

has this property.

The homogeneous part of the operator equation which corresponds to equation (12) is

$$(13) \qquad \sum_{\mu=0}^{M} \sum_{k=0}^{d} \sum_{\nu=0}^{N} \beta_{\mu,\nu}^{k} \mathbf{s}^{\nu} e^{-\tau_k s} \mathbf{x}^{(\mu)}(\lambda) = 0$$

where $M=m$, $N=\mathrm{Max}\,\{n_k\}$ and $\beta_{\mu,\nu}^{0} = \alpha_{\mu,\nu} \neq 0$, $\tau_0 = 0$.

The set of the points T for the equation (13) is $\{(\mu, 0)\}$, $\mu=0, \ldots, M$ and we have the third case of the proposition 1. This proposition asserts not only that the equaion (12) is logarithmic, mixed or pure simultaneously with its main part, but with the same method one can establish what of these three properties has it. When the condition $m \geqslant m_k$ is not satisfied, proposition 1 gives the answer about the number of linearly independent solutions of the equation (12) too.

5. Case when $(\tau_k^{\mu}) = (\alpha p_k^{\mu})$, p_k^{μ} an integer

In part 4 we analysed only the problem of the existence and the number of solutions of the equation (1) in general case. Now we shall restrict a little our equation (1) to obtain more precise results. First we suppose that the sequence $(\tau_k^{\mu}) = (\alpha p_k^{\mu})$, α positive number and p_k^{μ} and integer. The mathematical models we use frequently have an equation of the form (1) with $\mathbf{A}_{\mu}$ given by finite sums, so that we have only a finite number of τ_k^{μ}. If τ_k^{μ} are further rational numbers, it is easy to find our α. Also very often we have the case $\alpha = \pi$.

We shall suppose that α is unique for the equation (1). The finite numbers α does not give a really new situation. Finally we shall suppose that $p_k^{\mu} \geqslant 0$ which is only apparently a restriction.

Let us introduce the notation $e^{-\alpha s} = \mathbf{h}$, the equation (7) becomes

$$(14) \qquad \sum_{\mu=0}^{M} \sum_{k=0}^{\infty} \mathbf{a}_{\mu,k} \mathbf{h}^{p_k^{\mu}} \mathbf{w}^{\mu} = \sum_{\mu=0}^{M} \sum_{k=k_{\mu}}^{\infty} \mathbf{a}'_{\mu,k} \mathbf{h}^{k} \mathbf{w}^{\mu} = 0$$

where $k_{\mu} = p_0^{\mu}$, $a'_{\mu, p_0^{\mu}} = a_{\mu,0} \neq 0$.

Let us suppose that a solution of equation (14) is

$$w = \sum_{i=i_0}^{\infty} \mathbf{c}_i \mathbf{h}^{i/\beta} = \mathbf{h}^{i_0/\beta} \mathbf{v}, \quad \mathbf{c}_i \in \mathcal{A}, \quad \mathbf{v} \in \mathcal{H}_0. \tag{15}$$

Our proposition 1 we proved, says that $i_0 \geqslant 0$, $\mathbf{c}_{i_0} \neq 0$. The equation (14) has now the following form

$$\sum_{\mu=0}^{M} \sum_{k=k_\mu}^{\infty} \mathbf{a}'_{\mu,k} \, \mathbf{h}^{k+\frac{i_0}{\beta}\mu} \, \mathbf{v}^\mu = 0 \tag{16}$$

The rational number i_0/β can be found from the inequality

$$k_i + \left(\frac{i_0}{\beta}\right) i = k_j + \left(\frac{i_0}{\beta}\right) j \leqslant k_\mu + \left(\frac{i_0}{\beta}\right) \mu, \quad \mu = 0, 1, \ldots, M$$

which corresponds to inequality (10) and for $\mathbf{c}_{i_0}$ we have the equation which corresponds to equation (11).

With the notation $\mathbf{u} = \mathbf{h}^{1/\beta}$ equation (16) becomes

$$P(\mathbf{u}, \mathbf{v}) = \sum_{\mu=0}^{M} \sum_{k=k_\mu}^{\infty} \mathbf{a}'_{\mu,k} \, \mathbf{u}^{\beta k + i_0 \mu} \, \mathbf{v}^\mu = 0 \tag{17}$$

It is easy to see that $\mathbf{c}_{i_0}$ is a solution of $P(0, \mathbf{v}) = 0$. Let us suppose that $\mathbf{c}_{i_0}$ is not a multiple zero, we shall develop $P(\mathbf{u}, \mathbf{v})$ in powers of $\mathbf{u}$ and $\mathbf{v} - \mathbf{c}_{i_0}$. If this is not the case, we shall develop $P(\mathbf{u}, \mathbf{v})$ in powers of $\mathbf{v} - \mathbf{c}_{i_0} - \mathbf{c}_{i_0+1}\mathbf{u}$. It gives no difficulties because μ takes only values $0, 1, \ldots, M$. We have now

$$\mathbf{v} - \mathbf{c}_{i_0} = -\frac{1}{P_{0,1}(0, \mathbf{c}_{i_0})} \sum_{\mu=0}^{M} {\sum_{j=0}^{\infty}}' \frac{1}{\mu! \, k!} P_{j,\mu}(0, \mathbf{c}_{i_0}) \, \mathbf{u}^j (\mathbf{v} - \mathbf{c}_{i_0})^\mu \tag{18}$$

where $\sum'$ means that we omitted thet pair (0, 1) in the double sum.

From the equation (18) we can find all the coefficients $\mathbf{c}_i$ one after another taking

$$\mathbf{v} - \mathbf{c}_{i_0} = \sum_{i=1}^{\infty} \mathbf{c}_{i_0+i} \, \mathbf{u}^i$$

6. Case when the sequence $(\nu_{\mu,k})$ has a lower bound and $\sigma_{\mu,k}$ has an upper bound

We shall restrict a little more our equation (1). Let us suppose that there exist ν_0 and σ in such a way that for $\mu = 0, \ldots, M$ and $k \geqslant 0 : \nu_{\mu,k} \geqslant \nu_0$ and σ be the least common multiple for $\sigma_{\mu,k}$. For these reasons the coefficients $\mathbf{a}'_{\mu,k}$ can be written in the form:

$$\mathbf{a}'_{\mu,k} = \sum_{\nu=0}^{\infty} \beta_{\mu,\nu,k} \, l^{\nu/\sigma}$$

whence

$$\mathbf{A}_\mu = \sum_{k=0}^{\infty} \left(\sum_{\nu=0}^{\infty} \beta_{\mu,\nu,k}\, l^{\nu/\sigma} \right) \mathbf{h}^k$$

Let $\mathbf{f}_\mu$ be the convergence factor for $\mathbf{A}_\mu$, and $\mathbf{f}_{\mu,k}$ the convergence factor for $\mathbf{a}'_{\mu,k}$. For a fixed T we shall choose k' such that $\alpha k' < T$ and $\alpha(k'+1) \geqslant T$; with $\mathbf{F}_\mu = \mathbf{f}_\mu \prod_{k=0}^{k'} \mathbf{f}_{\mu,k}$ we have:

$$\mathbf{A}_\mu \mathbf{F}_\mu =_T \sum_{k=0}^{k'} \left(\sum_{\nu=0}^{\infty} \beta_{\mu,\nu,k}\, l^{\nu/\sigma} \right) \mathbf{h}^k\, \mathbf{F}_\mu$$

$$=_T \sum_{\nu=0}^{\infty} \left(\sum_{k=0}^{k'} \beta_{\mu,\nu,k}\, \mathbf{h}^k \right) l^{\nu/\sigma}\, \mathbf{F}_\mu$$

whence

$$\mathbf{A}_\mu \mathbf{F}_\mu = \sum_{\nu=0}^{\infty} \left(\sum_{k=0}^{\infty} \beta_{\mu,\nu,k}\, \mathbf{h}^k \right) l^{\nu/\sigma}\, \mathbf{F}_\mu .$$

From this relation follows that $\mathbf{A}_\mu$ can be taken as

(19) $$\mathbf{A}_\mu = \sum_{\nu=\nu_\mu}^{\infty} \mathbf{D}_{\mu,\nu}\, l^{\nu/\sigma}, \quad \mathbf{D}_{\mu,\nu_\mu} \neq 0, \quad \nu_\mu > 0$$

and

(20) $$\mathbf{D}_{\mu,\nu} = \sum_{k=k_{\mu,\nu}}^{\infty} \beta_{\mu,\nu,k}\, \mathbf{h}^k, \quad \beta_{\mu,\nu,k_{\mu,\nu}} \neq 0, \quad k_{\mu,\nu} > 0.$$

In the later we shall suppose that our $\mathbf{A}_\mu$ [from equation (1) has just the form given in (19).

In this part we shall leave alone the problems of the existence and the number of solutions of equation (1). Here we shall be employed with the character of the solution and its construction, if such solution exists.

First of all a lemma:

Lemma 2. *Let* $\{\mathbf{D}_{i_0+i}\} \subset \mathcal{H}(\alpha_i)$, $\sigma \in N$ *and the following series* $\sum_{i=1}^{\infty} \mathbf{D}_{i_0+i}\, l^{i/\sigma+1}$ *be convergent in* $\mathcal{C}$, *then*

$$\sum_{i=i_0}^{\infty} \mathbf{D}_i\, l^{i/\sigma} = 0 \Leftrightarrow \mathbf{D}_i = 0, \quad i = i_0,\ i_0+1, \ldots$$

Proof. Without any restriction we can start from

$$\sum_{i=i_0}^{\infty} \mathbf{D}_{i_0+i}\, l^{i/\sigma+1} = 0$$

whence

$$-\mathbf{D}_{i_0}\, l = \sum_{i=1}^{\infty} \mathbf{D}_{i_0+i}\, l^{i/\sigma+1}.$$

For $\mathbf{D}_{i_0} \neq 0$ this relation is not possible because $\mathbf{D}_{i_0}\, l$ is not continuous or is given by a constant and $\sum_{i=1}^{\infty} \mathbf{D}_{i_0+i}\, l^{i/\sigma+1}$ is a continuous function over every interval $[0, T]$, $T<\infty$ which is not a constant. In the same way we can prove that $\mathbf{D}_i = 0$, $i = i_0+1, \ldots$

The chracteristic equation of equation (1) with $\mathbf{A}_\mu$ given by (19) is

$$\sum_{\mu=0}^{M} \sum_{\nu=\nu_\mu}^{\infty} \mathbf{D}_{\mu,\nu}\, \mathbf{L}^\nu\, \mathbf{w}^\mu = 0 \tag{22}$$

where $\mathbf{L} = l^{1/\sigma}$.

Supposing a solution of this equation in the form

$$\mathbf{w} = \sum_{i=i_0}^{\infty} \mathbf{D}_i\, \mathbf{L}^{i/\gamma} = \mathbf{L}^{o/\gamma}\, \mathbf{V},\ \ \mathbf{D}_i \in \mathscr{H}(\delta_i),\ \ \delta_i = \alpha\, r_i,\ \ r_i \text{ rational},\ \gamma>0.$$

the characteristic equation (22) becomes

$$\sum_{\mu=0}^{M} \sum_{\nu=\nu_\mu}^{\infty} \mathbf{D}_{\mu,\nu}\, \mathbf{L}^{\nu+(i_0/\gamma)\mu}\, \mathbf{V}^\mu = 0.$$

To find the rational number i_0/γ, by lemma 2, we have an inequality

$$\nu_i + (i_0/\gamma)\, i = \nu_j + (i_0/\gamma)\, j \leqslant \nu_\mu + (i_0/\gamma)\, \mu,\ \ \mu = 0, \ldots, M \tag{23}$$

We know that this inequality has at least one solution and how it can be found. Let us suppose that the equality in (23) is obtained when μ takes the values $i>j>\cdots>t$, then to find $\mathbf{D}_{i_0}$ we have to solve the equation

$$\mathbf{D}_{i,\nu_i}\, \mathbf{D}_{i_0}^{i} + \mathbf{D}_{j,\nu_j}\, \mathbf{D}_{i_0}^{j} + \cdots + \mathbf{D}_{t,\nu_t}\, \mathbf{D}_{i_0}^{t} = 0, \tag{24}$$

We know that this equation has solutions because $\mathscr{H}$ is algebraically closed and this solution is of the form

$$\mathbf{D}_{i_0} = \sum_{j=j_0}^{\infty} \alpha_{i_0,j}\, \mathbf{h}^{\omega j},\ \ \omega \text{ is a rational number.}$$

The further mode of proceeding is the same as in 5. Let us suppose that $\mathbf{D}_{i_0}$ is a simple zero of equation (24). We shall introduce $\mathbf{U} = \mathbf{L}^{1/\gamma}$ and we shall have

$$P(\mathbf{U}, \mathbf{V}) = \sum_{\mu=0}^{M} \sum_{\nu=\nu_\mu}^{\infty} \mathbf{D}_{\mu,\nu}\, \mathbf{U}^{\nu\gamma+\mu i_0}\, \mathbf{V}^\mu = 0$$

whence

$$(25)\qquad \mathbf{V}-\mathbf{D}_{i_0}=\frac{-1}{P_{0,1}(0,\ \mathbf{D}_{i_0})}\sum_{\mu=0}^{M}{}' \sum_{j=0}^{\infty}\frac{1}{\mu!\,j!}P_{j,\mu}(0,\ \mathbf{D}_{i_0})\ \mathbf{U}^j(\mathbf{V}-\mathbf{D}_{i_0})^{\mu}$$

supposing that

$$(26)\qquad \mathbf{V}-\mathbf{D}_{i_0}=\sum_{i=1}^{\infty}\mathbf{D}_{i_0+i}\ \mathbf{U}^i$$

and comparing the powers of $\mathbf{U}$ in (25) we can obtain the coefficients $\mathbf{D}_{i_0+1}$, $\mathbf{D}_{i_0+2}, \ldots$ successively. Let us remember that we used the proposition of Lemma 1 and the fact all the coefficients are elements of the field $\mathcal{H}$.

7. Character of solutions of equation (1)

Let us suppose that we have found a solution

$$(28)\qquad \mathbf{w}=\sum_{i=i_0}^{\infty}\mathbf{D}_i\ \mathbf{U}^i=\sum_{i=i_0}^{\infty}\mathbf{D}_i\ l^{i/\gamma\sigma}$$

of equation (22). We discussed when such a solution is a logarithm. We ask the question now when a solution $\exp(\lambda\ \mathbf{w})$, $\mathbf{w}$ given by (28) is a function. To answer this question we shall use two lemmas:

Lemma 3. *Let $k_0 \geqslant 0$ and $\mathbf{D}=\sum_{k=k_0}^{\infty}\alpha_k\ \mathbf{h}^k \in \mathcal{H}(\alpha)$, then for every complex number $\lambda \neq 0$ $\exp(\lambda\ \mathbf{D})$ is a distribution which is not a function from $\mathcal{L}$,* $\exp(\lambda\ \mathbf{D})=\mathbf{s}\,\psi$, $\psi \in \mathcal{L}$.

Proof. — First we shall prove for $k_0 \geqslant 1$. Then we have $\mathbf{D}=\mathbf{h}^{k_0}\,\omega$, $\omega \in \mathcal{H}_0(\alpha)$ and

$$\exp(\lambda\ \mathbf{D})=\mathbf{s}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\mathbf{H}_{jk_0\alpha}\ \omega^j \equiv \mathbf{s}\ \psi.$$

For every $\lambda \neq 0$ ψ is a function from $\mathcal{L}$ which has $\alpha\, jk_0$, $j=1, 2, \ldots$ as points of discontinuity.

If $k_0=0$, then

$$\mathbf{D}=\alpha_0+\sum_{k=1}^{\infty}\alpha_k\ \mathbf{h}^k=\alpha_0+\mathbf{D}'$$

and $\exp(\lambda\ \mathbf{D})=\exp(\lambda\ \alpha_0)\exp(\lambda\ \mathbf{D}')$ which ends the proof.

Lemma 4. *Let $k_0 > 0$ and*

$$\mathbf{D}=\sum_{k=k_0}^{\infty}\alpha_k\ \mathbf{h}^k \in \mathcal{H}(\alpha)$$

then $\exp(\lambda \mathbf{s}^\nu \mathbf{D})$ *and* $\exp(\lambda \mathbf{s}^\nu \mathbf{D}) - \mathbf{I}$ *are not functions from* $\mathcal{L}$ *for every* $\nu > 0$ *and* λ *complex number* ($\mathbf{I}$ *is the unit in the field* $\mathcal{M}$), $\exp(\lambda \mathbf{s}^\nu \mathbf{D}) \times \mathbf{F}(0) \in \mathcal{C}$.

Proof. — We can write $\mathbf{D} = \mathbf{h}^{k_0} \omega$, $\omega \in \mathcal{H}_0(\alpha)$ and

$$\exp(\lambda \mathbf{s}^\nu \mathbf{h}^{k_0} \omega) = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \mathbf{s}^{\nu k+1} \mathbf{H}_{\alpha k_0 k} \omega^k \equiv \mathbf{I} + \psi$$

Let us suppose that ψ is a function from $\mathcal{L}$, then

$$\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \mathbf{F}(0) \mathbf{s}^{\nu k} \mathbf{h}^{k_0 k} \omega^k = \mathbf{F}(0) \psi$$

where $\mathbf{F}(0)$ is given in section 1. Let k' be chosen so that $\alpha k_0 k' < T$ and $\alpha k_0 (k'+1) \geqslant T$, for a fixed $T < \infty$:

$$\sum_{k=1}^{k'} \frac{\lambda^k}{k!} \mathbf{s}^{\nu k} \mathbf{F}(0) \mathbf{h}^{k_0 k} \omega^k =_T \mathbf{F}(0) \psi$$

$$\mathbf{F}(\nu k' + 1) \left\{ \sum_{k=1}^{k'} \frac{\lambda^k}{k!} l^{\nu k' - \nu k + 1} \mathbf{h}^{k_0 k} \omega^k - l^{\nu k' + 1} \psi =_T 0 \right.$$

By Titchmarsh's theorem [6] and the properties of the function $\mathbf{F}$ we have

$$\sum_{k=1}^{k'} \frac{\lambda^k}{k!} l^{\nu k' - \nu k + 1} \mathbf{h}^{k_0 k} \omega^k =_T l^{\nu k' + 1} \psi.$$

This can be written as

$$\frac{\lambda^{k'}}{k'!} \mathbf{H}_{\alpha k_0 k'} \alpha_{k_0}^{k'} + \aleph =_T l^{\nu k' + 1} \psi$$

where $\aleph$ is a continuous function in t in the neighbourhood of $t = \alpha k_0 k'$. We have now a contradiction; in the last relation the one side has a point of discontinuity $t = \alpha k_0 k' < T$ and the second one is continuous over $[0, T]$.

Let us suppose now that $\mathbf{I} + \psi$ is a function f from $\mathcal{L}$, then $\psi = f - \mathbf{I}$ and $\mathbf{F}(0)\ (f - \mathbf{I}) \in \mathcal{C}$, $l^{\nu k' + 1} (f - \mathbf{I}) \in \mathcal{C}$, so that the second part of the proof is the same as the first one.

From the properties of the function $\mathbf{F}(0)$ follows

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathbf{F}(0) \mathbf{s}^{\nu k} \mathbf{h}^{k_0 k} \omega^k \in \mathcal{C}$$

The character of the exponential operator $\exp(-\mu\, l^{-\alpha})$ is well known [1]:

Proposition D. 1. If $0<\alpha<1$ and $|\arg\mu|<\frac{\pi}{2}(1-\alpha)$

$$\exp(-\mu l^{-\alpha})=\begin{Bmatrix} t^{-1}\Phi(0,\ -\alpha;\ -\mu t^{-\alpha}), & t\neq 0 \\ 0 & t=0 \end{Bmatrix}$$

Φ is the function of E. M. Wright [5]. We see that our operator is from $\mathcal{C}$. For the special cases $\alpha=1/2$ and $\alpha=2/3$ see [5] pp. 115—116.

2. If $0<\alpha<1$ and $|\arg\mu|=\frac{\pi}{2}(1-\alpha)$.

$$\exp(-\mu l^{-\alpha})=\mathbf{s}l^{1/2}\{t^{-1/2}\Phi\left(\frac{1}{2},\ -\alpha;\ -|\mu|\exp\left[\pm\frac{\pi}{2}(1-\alpha)\right]it^{-\alpha}\right).$$

It is a distribution which is not a function from $\mathcal{L}$.

3. If $0<\alpha<1$ and $\pi\geqslant|\arg\mu|>\frac{\pi}{2}(1-\alpha)$, the expontial operator $\exp(-\mu l^{-\alpha})$ is an operator which is neither a function nor a distribution.

4. If $\alpha=1$ and $\mu>0$ $\exp(-\mu\mathbf{s})$ is the translation operator which corresponds to the $\delta(\mu)$ distribution.

5. For $\mathbf{f}\in\mathcal{L}$ we have $[\exp(\mathbf{f})-\mathbf{I}]\in\mathcal{L}$. In special case $\mathbf{f}=\mu l^{\alpha}$, $\alpha>0$, μ complex number it is $\exp(\mu l^{\alpha})=\mathbf{I}+\{t^{-1}\Phi(0,\alpha;\mu t^{\alpha})$. If $0<\alpha<1$, $[\exp(\mu l^{\alpha})-\mathbf{I}]\in\mathcal{L}$; for $\alpha\geqslant 1$ $\exp(\mu l^{\alpha})-\mathbf{I}$ is from $\mathcal{C}$.

Proposition 2. A solution $\exp(\lambda\mathbf{w})$, where $\mathbf{w}$ is given by (28) is a function in t for a fixed λ if and only if:

1. There exists at least one $i=i'\geqslant i_0$ such that

$$\beta_{i',0}\neq 0,\ |\arg(-\lambda\beta_{i',0})|<\frac{\pi}{2}(1+i'/(\gamma s)),\ -1<i'/(\gamma\sigma)<0.$$

2. $\beta_{i,0}=0$ for those $i<0$ for which $i/(\gamma s)<-1$.

Proof. — The solution $\mathbf{w}$ from (28) is of the form

$$w=\mathbf{D}_{i_0}l^{i_0/(\gamma s)}+\cdots+\mathbf{D}_{i'}l^{i'/(\gamma s)}+\cdots+\mathbf{D}_0+\mathbf{f}+\mathbf{g},$$

where $\mathbf{D}_i\in\mathcal{H}'$, $i=i_0,\ i_0+1,\ldots,$ $\mathbf{f}\in\mathcal{L}$ and $\mathbf{g}\in\mathcal{C}$; whence $[\exp\lambda\mathbf{f})-\mathbf{I}]\in\mathcal{L}$, $[\exp(\lambda\mathbf{g})-\mathbf{I}]\in\mathcal{C}$ and $l\exp(\lambda\mathbf{D}_0)\in\mathcal{L}$. Taking into account that $\mathbf{D}_i=\beta_{i,0}+\mathbf{D}'_i$, where $\mathbf{D}'_i=\sum_{j=1}^{\infty}\beta_{i,j}\mathbf{h}^j$ we have

$$\mathbf{w}=\sum_{i=i_0}^{-1}\mathbf{D}_i l^{i/(\gamma s)}+\mathbf{D}_0+\sum_{i=1}^{\frac{i}{\gamma s}<1}\mathbf{D}_i l^{i/(\gamma s)}+\sum_{\frac{i}{\gamma s}\geqslant 1}^{\infty}\mathbf{D}_i l^{i/(\gamma s)}$$

$$=\sum_{i=i_0}^{-1}\mathbf{D}_i l^{i/(\gamma s)}+\mathbf{D}_0+\mathbf{f}+\mathbf{g}.$$

From proposition B follows that $\mathbf{D}_i \in \mathcal{H}'$, $i = i_0, i_0 + 1, \ldots$ and from Proposition D we know that $[\exp \lambda \mathbf{f}) - \mathbf{I}] \in \mathcal{L}$ and $[\exp (\lambda \mathbf{g}) - \mathbf{I}] \in \mathcal{C}$. Lemma 3 gives that $\exp (\lambda \mathrm{D}_0) \in \mathbf{s}\, \mathcal{L}$. taking into account that $\mathbf{D}_i = \beta_{i,0} + \mathbf{D}'_i$ we have

$$\exp (\lambda \mathbf{w}) = \prod_{i=i_0}^{-1} \exp (\lambda \mathbf{D}_i\, l^{i/\gamma s}) \exp (\lambda \mathbf{D}_0) \exp (\lambda \mathbf{f}) \exp (\lambda \mathbf{g})$$

$$= \prod_{i=i_0}^{-1} \exp (\lambda \beta_{i,0}\, l^{i/\gamma s}) \prod_{i=i_0}^{-1} \exp (\lambda \mathbf{D}'_i\, l^{i/\gamma s}) \times$$

$$\times \exp (\lambda \mathbf{D}_0) \exp (\lambda \mathbf{f}) \exp (\lambda \mathbf{g}).$$

By Proposition D and Lemmas 3 and 4 only in the first product we can have a function in t for a fixed λ. The necessary and sufficient conditions for such a situation are given in our Proposition.

Using our Lemmas 3 and 4 and Proposition D we can say much more about the solutions $\exp(\lambda \mathbf{w})$, where $\mathbf{w}$ is given by (28). That means, is our solution from $\mathcal{C}^\infty$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{D}'$ or only from $\mathcal{M}$? Also we have the analytical form of this solutions The last two products are given by series in $l^{i/\gamma s}$. The third one is a sum of Heawiside functions on every interval $[0, T]$, $T < \infty$ Tha factors from the first product are given by Proposition D and we see that all are expressed by Wright's function. The factors from the second product are given by Lemma 4.

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