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ON SOME EXTREME PROPERTIES OF A CLASS OF SIMPLE GROUPOIDS

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Abstract **We introduce an easy method of constructing a simple groupoid which contains a given groupoid as subgroupoid, and show that the resulting simple groupoid admits only discrete topology and trivial partially ordered relation.**

In [2], J. Ježek has constructed a full embedding of the category of monomorphisms of groupoids into the category of homomorphisms of simple groupoids without idempotents. In a crucial step of his construction, he has shown that every groupoid can be embedded into a simple groupoid (i.e. groupoid which has only two trivial congruences). The simple groupoid constructed by Ježek is quite complicated. Even for a finite groupoid, the resulting simple groupoid contains infinitely many elements.

In [3], we have shown that every groupoid has at least countably infinite ways of embedding into a simple groupoid. The simplest way is given as follows:

Let $\langle G, \circ \rangle$ be a given groupoid. To each $a \in G$, we associate an element $\bar{a}$ and let $\overline{G} = \{\bar{a} \mid a \in G\}$. Assuming that $G \cap \overline{G} = \varnothing$, we set $\tilde{G} = G \cup \overline{G} \cup \{0\}$ where $0 \notin \overline{G} \cup G$. Define a multiplication on $\tilde{G}$ as follows:

(1) $0 \cdot 0 = 0$

(2) $\bar{a} \cdot \bar{a} = \bar{a}$ for every $\bar{a} \in \overline{G}$.

(3) $0 \cdot a = a \cdot 0 = \bar{a}$ for every $a \in G$.

(4) $0 \cdot \bar{a} = \bar{a} \cdot 0 = a$ for every $\bar{a} \in \overline{G}$.

(5) $a \cdot \bar{a} = \bar{a} \cdot a = 0$ for every $a \in G$.

(6) $a \cdot b = a \circ b$ for every $a,\ b \in G$.

(7) $\bar{a} \cdot b = b \cdot \bar{a} = \bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a} = 0$ for every $\bar{a},\ \bar{b} \in \overline{G},\ \bar{a} \neq \bar{b}$, and for every $b \in G$.

Theorem 1. *$\langle \overline{G}, \circ \rangle$ is a simple groupoid containing G as a subgroupoid.*

Proof. Let θ be a non-trivial congruence of $\tilde{G}$. Then there exist x, $y \in \tilde{G}$, $x \neq y$ such that $x \ \theta \ y$. Consider the following possible cases:

case 1 $x=0$ and $y=a$ for some $a \in G$.

$$\begin{aligned} x \theta y &\Rightarrow 0 \theta a \\ &\Rightarrow 0 \cdot 0 \theta 0 \cdot a \\ &\Rightarrow 0 \theta \bar{a} \\ &\Rightarrow 0 \cdot b \theta \bar{a} \cdot b \quad \text{for every } b \in G. \\ &\Rightarrow \bar{b} \theta 0 \\ &\Rightarrow 0 \cdot \bar{b} \theta 0 \cdot 0 \\ &\Rightarrow b \theta 0 \\ &\Rightarrow \theta = \tilde{G} \times \tilde{G}. \end{aligned}$$

case 2 $x=0$ and $y=\bar{a}$ for some $\bar{a} \in \overline{G}$.

By the same argument, $\theta = \tilde{G} \times \overline{G}$.

case 3 $x=a$ and $y=\bar{b}$ for some $a \in G$, $\bar{b} \in \overline{G}$.

$$\begin{aligned} x \theta y &\Rightarrow a \theta \bar{b} \\ &\Rightarrow a \cdot \bar{b} \theta \bar{b} \cdot \bar{b} \\ &\Rightarrow 0 \theta \bar{b} \quad \text{which reduces to case 2.} \end{aligned}$$

case 4 $x=\bar{a}$ and $y=\bar{b}$ for some $\bar{a}$, $\bar{b} \in \overline{G}$.

$$\begin{aligned} x \theta y &\Rightarrow \bar{a} \theta \bar{b} \\ &\Rightarrow \bar{a} \cdot \bar{a} \theta \bar{a} \cdot \bar{b} \\ &\Rightarrow \bar{a} \theta 0 \quad \text{which also reduces to case 2.} \end{aligned}$$

case 5 $x=a$ and $y=b$ for some a, $b \in G$.

$$\begin{aligned} x \theta y &\Rightarrow a \theta b \\ &\Rightarrow 0 \cdot a \theta 0 \cdot b \\ &\Rightarrow \bar{a} \theta \bar{b} \quad \text{which reduces to case 4.} \end{aligned}$$

Hence $\langle \tilde{G}, \cdot \rangle$ is a simple groupoid which contains G as a subgroupoid.

A topological groupoid $\langle G, \cdot, \tau \rangle$ is a system consisting of a set G on which a binary operation $\cdot$ is defined, and a Hausdorff topology τ such that the operation $\cdot$ is continuous with respect to τ.

Theorem 2. *$\langle \tilde{G}, \cdot \rangle$ admits only discrete topology.*

Proof. Let τ be a Hausdorff topology on $\tilde{G}$ such that the multiplication $\cdot$ is continuous.

Let a be any element of G. We have $0\cdot\bar{a}=a$. Let W be an open neighbourhood of a which does not contain 0. Then there exist open neighbourhoods U and V such that $0\in U$, $\bar{a}\in V$ and $U\cdot V\subset W$. By (5) and (7), $0\in U\subset\{0, \bar{a}\}$, so that $\{0\}=U-\{\bar{a}\}$ is open.

Again, for every $\bar{a}\in\overline{G}$, we have $\bar{a}\cdot\bar{a}=\bar{a}$. Let W be and open neighbourhood of $\bar{a}$ which does not contain 0. Then there exist open neighbourhoods U and V of $\bar{a}$ such that $U\cdot V\subset W$. By (5) and (7), we have $\bar{a}\in U\subset\{0, \bar{a}\}$ so that $\{\bar{a}\}$ is open.

Finally, we have $0\cdot a=\bar{a}$ for every $a\in G$. Take $W=\{\bar{a}\}$ which we have already proved to be open. There exist open neighbourhoods U and V such that $0\in U$, $a\in V$ and $U\cdot V\subset W$. This implies that $U=\{0\}$, $V=\{a\}$ is that $\{a\}$ is open.

Hence τ is the discrete topology.

We recall that a system $\langle G, \cdot, \leqslant\rangle$ consisting of a groupoid $\langle G, \cdot\rangle$ and a partially ordered relation $\leqslant$ is a partially ordered groupoid if for each $a\in G$ and for $b\leqslant c$ in G, we always have $a\cdot b\leqslant a\cdot c$, $b\cdot a\leqslant c\cdot a$.

Theorem. 3. *$\langle\tilde{G}, \cdot\rangle$ admits only trivial partially ordered relation* $\leqslant=\{x, x \mid x\in\tilde{G}\}$.

Proof. Let $\leqslant$ be a partially ordered relation on $\tilde{G}$. Consider the following possible cases:

case (i) If $0\leqslant a$ for some $a\in G$, we have

$$0\cdot\bar{a}\leqslant a\cdot\bar{a}$$
$$\Rightarrow a\leqslant 0$$
$$\Rightarrow 0=a \text{ which is a contradiction.}$$

case (ii) Similarly, if $0\leqslant\bar{a}$ for some $\bar{a}\in\overline{G}$, we have $\bar{a}=0$ which is a contradiction.

case (iii) If $\bar{a}\leqslant\bar{b}$ for some $\bar{a}$, $\bar{b}\in\overline{G}$, $\bar{a}\neq\bar{b}$, we have

$$\bar{a}\cdot\bar{a}\leqslant\bar{a}\cdot\bar{b}$$
$$\Rightarrow \bar{a}\leqslant 0 \text{ which reduces to case (ii).}$$

case (iv) If $a\leqslant b$ for some a, $b\in G$, $a\neq b$, we obtain

$$0\cdot a\leqslant 0\cdot b$$
$$\Rightarrow \bar{a}\leqslant\bar{b} \text{ which reduces to case (iii).}$$

case (v) If $a\leqslant\bar{b}$ for some a, $b\in G$, we have

$$a\cdot\bar{b}\leqslant\bar{b}\cdot\bar{b}$$
$$\Rightarrow 0\leqslant\bar{b} \text{ which reduces to case (ii).}$$

Hence we conclude that $\leqslant=\{(x, x) \mid x\in\tilde{G}\}$.

Corollary. *Every groupoid can be embedded into a simple groupoid which admits only discrete topology and trivial partially ordered relation.*

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