PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 25 (39), 1979, pp. 167—178

ON A HYPOTHESIS OF D. ADAMOVIĆ CONCERNING ASYMPTOTIC BEHAVIOUR OF SOME COMPLEX SEQUENCES

Slavko Simić

(Received October 6, 1978)

0. D. Adamović has formulated in [2] the following.

Hypothesis. *Let*

(1) $$(\alpha_n)_{n \in N_0}$$

($N_0 \overset{\text{def}}{=} N \cup \{0\}$) *be a sequence of real numbers.*

1° *If the sequence* (1) *is strictly increasing for* $n \geqslant 0$, $\lim\limits_{n \to +\infty} \alpha_n = +\infty$ *and* $\lim\limits_{n \to +\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_{n+1} - \alpha_{n+2}} = 1$, *then for any* $\theta \in (0, 2\pi)$

$$\sum_{k=0}^{n} e^{k\theta i} \alpha_k \sim \frac{e^{\theta i}}{e^{\theta i} - 1} e^{n\theta i} \alpha_n \qquad (n \to +\infty).$$

2° *If* (1) *is strictly decreasing for* $n \geqslant 0$, $\lim\limits_{n \to +\infty} \alpha_n = 0$ *and* $\lim\limits_{n \to +\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1} - \alpha_{n+2}} = 1$, *then*

$$\sum_{k=n}^{+\infty} e^{k\theta i} a_k \sim \frac{1}{1 - e^{\theta i}} e^{n\theta i} \alpha_n \qquad (n \to +\infty).$$

In this note, among the other results, we shall prove that the previous hypothesis is true.

0.1. In the proof of this fact and in our other considerations we shall use the following:

A. *Let* (a_n) *be an arbitrary sequence of real numbers and* (B_n) *such a sequence which strictly increases and tends to* $+\infty$. *Then for* l *real* $l = +\infty$ *and* $l = -\infty$,

$$\lim_{n \to +\infty} \frac{A_n - A_{n-1}}{B_n - B_{n\ 1}} = l \Rightarrow \lim_{n \to +\infty} \frac{A_n}{B_n} = l,$$

while the inverse implication is not true.

B. *If* (A_n) *and* (B_n) *are zero-sequences of real numbers,* (B_n) *being strictly increasing, then both assertions in* **A** *hold.*

$$\text{C. If } t_n=\sum_{k=0}^{n} e^{k\theta i}, \text{ then } |t_n|\leqslant\frac{1}{\sin\frac{\theta}{2}} \text{ for } \theta\in(0,2\pi).$$

A. is known as Stolz's theorem, **B** was proved, for instance, in [3] (p. 127—128) and the fact **C** is simple and well-known.

1. Let us remark that the conditions of Hypothesis can be formulated somewhat differently (and more generally). Namely, other conditions in 1° and 2° imply both corresponding monotonies, for n sufficiently large and the validity of Hypothesis under formulated conditions imply, obviously, its validity if these conditions hold only for n large enough. For this reason, we formulate Hypothesis as follows:

Theorem 1. *Let* (α_n) *be a sequence of real numbers.*

1° *If*

$$\lim_{n\to+\infty}\alpha_n=+\infty \quad \text{and} \quad \lim_{n\to+\infty}\frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+2}-\alpha_{n+1}}=1, \tag{2}$$

then for each $\theta\in(0,2\pi)$

$$\sum_{k=0}^{n} e^{k\theta i}\alpha_k\sim\frac{e^{\theta i}}{e^{\theta i}-1}e^{n\theta i}\alpha_n \qquad (n\to+\infty).$$

2° *If*

$$\lim_{n\to+\infty}\alpha_n=0 \quad \text{and} \quad \lim_{n\to+\infty}\frac{\alpha_n-\alpha_{n+1}}{\alpha_{n+1}-\alpha_{n+2}}=1, \tag{3}$$

then for each $\theta\in(0,2\pi)$

$$\sum_{k=n}^{+\infty} e^{k\theta i}\alpha_k\sim\frac{1}{1-e^{\theta i}}e^{n\theta i}\alpha_n \qquad (n\to+\infty).$$

Proof. First we remark that conditions of the theorem imply, in both cases, strict monotony of sequence (α_n), and also in the case 1°

$$\alpha_n>0 \text{ for } n \text{ sufficiently large,} \tag{4}$$

and in the case 2°, (4) or (4) with $<$ instead of $>$. The second possibility being easily reductible to the first one, we will also suppose the condition (4) satisfied in the case 2°. Further, by **A** and **B**, in both cases

$$\lim_{n\to+\infty}\frac{\alpha_{n+1}}{\alpha_n}=1. \tag{5}$$

We will consider each of two cases separately.

1° The application of Abel's formula of partial summation gives, for $\theta\in(0,2\pi)$ and $n\geqslant 1$,

$$\sum_{k=0}^{n} e^{k\theta i}\alpha_k-\frac{e^{\theta i}}{e^{\theta i}-1}e^{n\theta i}\alpha_n=$$

$$\sum_{k=0}^{n-1}\frac{e^{(k+1)\theta i}-1}{e^{\theta i}-1}(\alpha_k-\alpha_{k+1})+\frac{e^{(n+1)\theta i}-1}{e^{\theta i}-1}\alpha_n-\frac{e^{(n+1)\theta i}}{e^{\theta i}-1}\alpha_n=$$

$$=\frac{1}{e^{\theta i}-1}\left[\sum_{k=0}^{n-1}e^{(k+1)\theta i}(\alpha_k-\alpha_{k+1})-\alpha_0+\alpha_n-\alpha_0\right],$$

and so

$$\sum_{k=0}^{n} e^{k\theta i}\alpha_k-\frac{e^{\theta i}}{e^{\theta i}-1}e^{n\theta i}\alpha_n=\frac{1}{e^{\theta i}-1}\left[-\alpha_0+\sum_{k=0}^{n-1}e^{(k+1)\theta i}(\alpha_k-\alpha_{k+1})\right]\quad(n\geqslant 2).\tag{6}$$

Applying Abel's partial summation again, we get further

$$\frac{1}{e^{\theta i}-1}=\left[-\alpha_0+\sum_{k=0}^{n-1}e^{(k+1)\theta i}(\alpha_k-\alpha_{k+1})\right]=$$

$$\frac{1}{e^{\theta i}-1}\left[-\alpha_0+\sum_{k=0}^{n-1}(t_k-t_{k+1})(\alpha_k-\alpha_{k+1})\right]=$$

$$\frac{1}{e^{\theta i}-1}\left[-\alpha_0+\sum_{k=0}^{n-1}(t_{k+1}-1)(\alpha_k-2\alpha_{k+1}+\alpha_{k+2})+(t_n-1)(\alpha_{n-1}-\alpha_n)\right].$$

So we have

$$\begin{cases}\displaystyle\sum_{k=0}^{n} e^{k\theta i}\alpha_k-\frac{e^{\theta i}}{e^{\theta i}-1}e^{n\theta i}\alpha_n=\\ \displaystyle\frac{1}{e^{\theta i}-1}\left[-2\alpha_0+\alpha_1+t_n(\alpha_n-\alpha_{n-1})+\sum_{k=0}^{n-2}t_{k+1}(\alpha_k-2\alpha_{k+1}+\alpha_{k+2})\right]\\ \qquad\qquad\qquad\qquad(0<\theta<2\pi;\ n\geqslant 2).\end{cases}\tag{7}$$

From this and accordingly to **C**, we obtain for n sufficiently large

$$\Delta_n\overset{\text{def}}{=}\left|\frac{\sum_{k=0}^{n}e^{\theta k i}\alpha_k}{\frac{e^{\theta i}}{e^{\theta i}-1}e^{n\theta i}\alpha_n}-1\right|\leqslant$$

$$\leqslant\frac{|-2\alpha_0+\alpha_1|}{\alpha_n}+\frac{1}{\sin\frac{\theta}{2}}\left|\frac{\alpha_{n-1}}{\alpha_n}-1\right|+\frac{1}{\sin\frac{\theta}{2}}\cdot\frac{\sum_{k=0}^{n-2}|\alpha_k-2\alpha_{k+1}+\alpha_{k+2}|}{\alpha_n}$$

and further, applying **A** and using (2) and (4),

$$\varlimsup_{n\to+\infty}\Delta_n\leqslant|-2\alpha_0+\alpha_1|\lim_{n\to+\infty}\frac{1}{\alpha_n}+\frac{1}{\sin\frac{\theta}{2}}\left|\lim_{n\to+\infty}\frac{\alpha_{n-1}}{\alpha_n}-1\right|+$$

$$+\frac{1}{\sin\frac{\theta}{2}}\lim_{n\to+\infty}\frac{|\alpha_{n-2}-2\alpha_{n-1}+\alpha_n|}{\alpha_n-\alpha_{n-1}}=$$

$$=\frac{1}{\sin\frac{\theta}{2}}\lim_{n\to+\infty}\left|1-\frac{\alpha_{n-1}-\alpha_{n-1}}{\alpha_n-\alpha_{n-1}}\right|=\frac{1}{\sin\frac{\theta}{2}}\left|1-\lim_{n\to+\infty}\frac{\alpha_{n-1}-\alpha_{n-2}}{\alpha_n-\alpha_{n-1}}\right|=0.$$

Hence, $\lim_{n\to+\infty}\Delta_n=0$, which means that

$$\sum_{k=0}^{n}e^{k\theta i}\alpha_n\sim\frac{e^{\theta i}}{e^{\theta i}-1}e^{n\theta i}\alpha_n \qquad (n\to+\infty).$$

2° As we have said, we can suppose that the condition (4) is satisfied in this case, and then (1) is strictly decreasing for n sufficiently large. Then the series $\sum_{k=0}^{\infty}e^{k\theta i}\alpha_k$ $(0<\theta<2\pi)$, as is known, converges, and also the series $\sum_{k=0}^{+\infty}e^{k\theta i}(\alpha_k-\alpha_{k+1})$, $\sum_{k=0}^{+\infty}t_k(\alpha_{k+1}-\alpha_k)$ and $\sum_{k=0}^{\infty}t_{k+1}(\alpha_{k+1}-\alpha_k)$ $(0<\theta<2\pi)$, since

$$|e^{k\theta i}(\alpha_k-\alpha_{k+1})|\leqslant\alpha_k-\alpha_{k+1},\quad |t_k(\alpha_{k+1}-\alpha_k)|\leqslant\frac{1}{\sin\frac{\theta}{2}}(\alpha_k-\alpha_{k+1}),$$

$$|t_{k+1}(\alpha_{k+1}-\alpha_k)|\leqslant\frac{1}{\sin\frac{\theta}{2}}|\alpha_k-\alpha_{k+1}|,$$

for k large enough, and

$$\sum_{k=0}^{n}(\alpha_k-\alpha_{k+1})=\alpha_0-\alpha_{n+1}\to\alpha_0 \qquad (n\to+\infty).$$

We have

$$\sum_{k=n}^{+\infty}e^{k\theta i}\alpha_k-\frac{1}{1-e^{\theta i}}e^{n\theta i}\alpha_n=\frac{1}{1-e^{\theta i}}\left\{\sum_{k=0}^{+\infty}[e^{k\theta i}-e^{(k+1)\theta i}]\alpha_k-e^{n\theta i}\alpha_n\right\}=$$

$$\frac{1}{1-e^{\theta i}}\left[\sum_{k=n+1}^{+\infty}e^{k\theta i}\alpha_k-\sum_{k=n+1}^{+\infty}e^{(k+1)\theta i}\alpha_k\right]=\frac{1}{1-e^{\theta i}}\sum_{k=n}^{+\infty}e^{(k+1)\theta i}(\alpha_{k+1}-\alpha_k),$$

such that

$$\sum_{k=n}^{+\infty} e^{k\theta i}\alpha_k - \frac{1}{e^{\theta i}-1} e^{n\theta i}\alpha_n = \frac{1}{1-e^{\theta i}}\sum_{k=n}^{+\infty} e^{(k+1)\theta i}(\alpha_{k+1}-\alpha_k) \quad (0<\theta<2\pi;\ n\geqslant 0). \tag{8}$$

Further, the right hand side is equal to

$$\frac{1}{1-e^{\theta i}}\sum_{k=n}^{+\infty}(t_{k+1}-t_k)(\alpha_{k+1}-\alpha_k)=$$

$$\frac{1}{1-e^{\theta i}}\left[\sum_{k=n}^{+\infty} t_{k+1}(\alpha_{k+1}-\alpha_k)-\sum_{k=n}^{+\infty} t_k(\alpha_{k+1}-\alpha_k)\right]$$

and so

$$\sum_{k=n}^{+\infty} e^{k\theta i}\alpha_k - \frac{1}{1-e^{\theta i}} e^{n\theta i}\alpha_n = \tag{9}$$

$$\frac{1}{1-e^{\theta i}}\left[t_n\alpha_n-\alpha_{n+1})-\sum_{k=n+1}^{+\infty} t_k(\alpha_{k-1}-2\alpha_k+\alpha_{k-1})\right] \quad (0<\theta<2\pi;\ n\geqslant 0).$$

Hence

$$\nabla_n \stackrel{\text{def}}{=} \left|\frac{\sum_{k=n}^{+\infty} e^{k\theta i}\alpha_k}{\frac{1}{1-e^{\theta i}}e^{n\theta i}\alpha_n}-1\right|$$

$$\leqslant \frac{1}{\sin\frac{\theta}{2}}\left(1-\frac{\alpha_{n+1}}{\alpha_n}+\frac{\sum_{k=n+1}^{+\infty}|\alpha_{k-1}-2\alpha_k+\alpha_{k+1}|}{\alpha_n}\right).$$

Letting $n\to+\infty$, one obtains, by **B** and (5),

$$\overline{\lim_{n\to+\infty}}\nabla_n \leqslant \frac{1}{\sin\frac{\theta}{2}}\left(1-\lim_{n\to+\infty}\frac{\alpha_{n+1}}{\alpha_n}+\lim_{n\to+\infty}\frac{\sum_{k=n+1}^{+\infty}|\alpha_{k-1}-2\alpha_k+\alpha_{k-1}|}{\alpha_n}\right)=$$

$$\frac{1}{\sin\frac{\theta}{2}}\lim_{n\to+\infty}\frac{|\alpha_n-2\alpha_{n+1}+\alpha_{n+2}|}{\alpha_n-\alpha_{n+1}}=\frac{1}{\sin\frac{\theta}{2}}\lim_{n\to+\infty}\left|1-\frac{\alpha_{n+1}-\alpha_{n+2}}{\alpha_n-\alpha_{n+1}}\right|=0,$$

which implies $\lim_{n\to+\infty}\nabla_n=0$, that is

$$\sum_{k=n}^{+\infty} e^{k\theta i}\alpha_k \sim \frac{1}{1-e^{\theta i}}e^{n\theta i}\alpha_n \qquad (n\to+\infty).$$

2. Supplementary results

We add to the previous result some analoguous and in a certain senses supplementary results, else incomparable to it. We formulate them as Theorems 2—4. The conditions of each of these theorems contain some supposition on convexity or concavity of the sequence (1) of real numbers

Theorem 2. *If for* $n \geqslant 0$

$$\alpha_{n+1} - \alpha_n \geqslant 0 \wedge \alpha_{n+1} - 2\alpha_n + \alpha_{n-1} \geqslant 0, \tag{10}$$

then for each $\theta \in (0, 2\pi)$

$$\sum_{k=0}^{n} e^{k\theta i} \alpha_k = \frac{e^{(n+1)\theta i} \alpha_n - \alpha_0}{e^{\theta i} - 1} + 0(\alpha_n - \alpha_{n-1}) \qquad (n \to +\infty).$$

Proof. By (7) and (10), we have for $\theta \in (0, 2\pi)$

$$\left| \sum_{k=0}^{n} e^{k\theta i} \alpha_k - \frac{e^{(n+1)\theta i} - \alpha_0}{e^{\theta i} - 1} \right| =$$

$$\frac{1}{|e^{\theta i} - 1|} \left| t_n (\alpha_{n-1} - \alpha_n) + \alpha_1 - \alpha_0 + \sum_{k=0}^{n-2} t_{k+1} (\alpha_k - 2\alpha_{k+1} + \alpha_{k+2}) \right| \cdot$$

$$\leqslant \frac{1}{2\sin\frac{\theta}{2}} \left[|t_n| (\alpha_n - \alpha_{n-1}) + \alpha_1 - \alpha_0 + \sum_{k=0}^{n-2} |t_{k+1}| (\alpha_k - 2\alpha_{k+1} + \alpha_{k+2}) \right]$$

$$\leqslant \frac{1}{2\sin^2\frac{\theta}{2}} \left\{ \alpha_n - \alpha_{n-1} + \alpha_1 - \alpha_0 + \sum_{k=0}^{n-2} [\alpha_{k+2} - \alpha_{k+1} - (\alpha_{k+1} - \alpha_k)] \right\}$$

$$= \frac{1}{2\sin^2\frac{\theta}{2}} [\alpha_n - \alpha_{n-1} + \alpha_1 - \alpha_0 + \alpha_n - \alpha_{n-1} - (\alpha_1 - \alpha_0)]$$

$$= \frac{1}{2\sin^2\frac{\theta}{2}} (\alpha_n - \alpha_{n-1}) = 0(\alpha_n - \alpha_{n-1}) \qquad (n \to +\infty).$$

Corollary 1. *If the sequence* (α_n) *satisfies the condition of Theorem* 2 *and in addition* $\lim_{n\to+\infty} \alpha_n = +\infty$ *and* $\lim_{n\to+\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$, *then for* $\theta \in (0, 2\pi)$

$$\sum_{k=0}^{n} e^{k\theta i} \alpha_k \sim \frac{e^{\theta i}}{e^{\theta i} - 1} e^{n\theta i} \alpha_n \qquad (n \to +\infty).$$

Remark 1. It is easy to see that if (α_n) satisfies (10), even for n sufficiently large only, then $\lim_{n\to+\infty} \alpha_n = +\infty$, or α_n is constant for n large enough. Excluding the last (trivial) case in Theorem 2, one can, obviously, replace the condition of Theorem 2, and at the same time the first condition of Corollary 1, by the following weaker condition (10) for n sufficiently large.

Theorem 3. *Let, for n large enough,*

$$\alpha_n \geqslant 0 \wedge \alpha_{n+1} - \alpha_n \geqslant 0 \wedge \alpha_{n+1} - 2\alpha_n + \alpha_{n-1} \leqslant 0. \tag{11}$$

Then, for each $\theta \in (0, 2\pi)$:

$$\left.\begin{aligned} &1^\circ \sum_{k=0}^{n} e^{k\theta i} \alpha_k = \frac{e^{(n+1)\theta i} \alpha_n - T - \alpha_0}{e^{\theta i} - 1} + 0(\alpha_{n+1} - \alpha_n), \\ &\qquad \text{with } T = \sum_{k=0}^{+\infty} e^{(k+1)\theta i} (\alpha_{k+1} - \alpha_k) \end{aligned}\right\} \text{ if } \alpha \overset{\text{def}}{=} \lim_{n\to+\infty} \alpha_n \in [0, +\infty);$$

$$2^\circ \sum_{k=0}^{n} e^{k\theta i} \alpha_k = \frac{e^{\theta i}}{e^{\theta i} - 1} e^{n\theta i} \alpha_n + 0(1) \sim \frac{e^{\theta i}}{e^{\theta i} - 1} e^{n\theta i} \alpha_n \qquad (n \to +\infty),$$

$$\text{if } \alpha = +\infty.$$

Proof. *Case* 1°. In this case the series $\sum_{k=0}^{+\infty} e^{(k+1)\theta i} (\alpha_{k+1} - \alpha_k)$, $\sum_{k=0}^{+\infty} t_k (\alpha_{k+1} - \alpha_k)$ and $\sum_{k=0}^{+\infty} t_{k+1} (\alpha_{k+1} - \alpha_k)$ $(0 < \theta < 2\pi)$ converge (absolutely), which can be established similarly as in the proof of Theorem 1. Then, in view of (6), (8) and (9), we have for $\theta \in (0, 2\pi)$

$$\left| \sum_{k=0}^{n} e^{k\theta i} \alpha_k - \frac{e^{(n+1)\theta i} \alpha_n - T - \alpha_0}{e^{\theta i} - 1} \right| =$$

$$\frac{1}{|e^{\theta i} - 1|} \left| T - \sum_{k=0}^{n-1} e^{(k+1)\theta i} (\alpha_{k+1} - \alpha_k) \right| = \frac{1}{|e^{\theta i} - 1|} \left| \sum_{k=n}^{+\infty} e^{(k+1)\theta i} (\alpha_{k+1} - \alpha_k) \right| =$$

$$\frac{1}{|e^{\theta i} - 1|} \left| t_n (\alpha_n - \alpha_{n+1}) - \sum_{k=n+1}^{+\infty} t_k (\alpha_{k-1} - 2\alpha_k + \alpha_{k+1}) \right| \leqslant$$

$$\frac{1}{2\sin^2 \frac{\theta}{2}} \left\{ a_{n+1} - \alpha_n + \sum_{k=n+1}^{+\infty} [(\alpha_k - \alpha_{k+1}) - (\alpha_{k+1} - \alpha_k)] \right\} =$$

$$\frac{1}{2\sin^2 \frac{\theta}{2}} [2(\alpha_{n+1} - \alpha_n) - \lim_{n\to+\infty} (\alpha_{p+1} - \alpha_p)] =$$

$$\frac{\alpha_{n+1} - \alpha_n}{\sin^2 \frac{\theta}{2}} = 0(\alpha_{n+1} - \alpha_n) \qquad (n \to +\infty).$$

Case 2°. Under the supposition that (11) holds for $n\geqslant 0$, using (7), for $\theta\in(0,2\pi)$ we obtain

$$\left|\sum_{k=0}^{n} e^{k\theta i}\alpha_n-\frac{e^{(n+1)\theta i}\alpha_n}{e^{\theta i}-1}\right|=$$

$$\frac{1}{|e^{\theta i}-1|}\left|-2\alpha_0+\alpha_1+t_n(\alpha_{n+1}-\alpha_n)+\sum_{k=0}^{n-2}t_{k+1}(\alpha_k-2\alpha_{k+1}+\alpha_{k+2})\right|\leqslant$$

$$\frac{1}{2\sin\frac{\theta}{2}}\left\{|-2\alpha_0+\alpha_1|+\alpha_{n+1}-\alpha_n+\sum_{k=0}^{n-2}[\alpha_{k+1}-\alpha_k-(\alpha_{k+2}-\alpha_{k+1})]\right\}=$$

$$\frac{1}{2\sin\frac{\theta}{2}}[|-2\alpha_0+\alpha_1|+\alpha_1-\alpha_0+\alpha_{n+1}-2\alpha_n+\alpha_{n-1}]\leqslant$$

$$\frac{1}{2\sin\frac{\theta}{2}}[2(\alpha_1-\alpha_0)+\alpha_0]=0\,(1)\qquad(n\to+\infty).$$

If the condition (11) is satisfied only for n sufficiently large, then 2° obviously holds too.

Remark 2. Let us put

$$w=\frac{e^{\theta i}}{e^{\theta i}-1}\qquad(0<\theta<2\pi).$$

Then $\operatorname{Re} w=\frac{1}{2}$ and, by the assertion 2° of Theorem 3, under the condition

(12) $\quad(\alpha_{n+1}-\alpha_n\geqslant 0\wedge\alpha_{n+1}-2\alpha_n+\alpha_{n-1}\leqslant 0)$ for n enough $\wedge\lim\limits_{n\to+\infty}\alpha_n=+\infty$

holds

(13) $\quad\sum\limits_{k=0}^{n}a_k\sim wa_n\qquad(n\to+\infty)$, with $a_n=\alpha_n e^{n\theta i}\qquad(0<\theta<2\pi;\ n\geqslant 0)$.

This implies, by [1] (Teorem 3), that then the necessary condition for (14)

$$e^{\theta i}=\frac{w}{w-1}=\lim_{n\to+\infty}\frac{a_{n+1}}{\alpha_n}=e^{\theta i}\lim_{n\to+\infty}\frac{\alpha_{n+1}}{\alpha_n}$$

is satisfied, i.e. that

(14) $$\lim_{n\to+\infty}\frac{\alpha_{n+1}}{\alpha_n}=1.$$

So we have obtained the following result; (13) *implies* (14).

Of course, this assertion can be easily proved in a direct manner [as follows: (12) implies first that $\Delta\alpha_n\overset{\text{def}}{=}\alpha_{n+1}-\alpha_n$ is nonnegative and decreasing for n large enough, and consequently $\lambda\overset{\text{def}}{=}\lim\limits_{n\to+\infty}\Delta\alpha_n$ exists and $\lambda\in[0,+\infty)$ hence $\lim\limits_{n\to+\infty}\left(\frac{\alpha_{n+1}}{\alpha_n}-1\right)=\lim\limits_{n\to+\infty}\frac{\Delta\alpha_n}{\alpha_n}=0$].

Theorem 4. 1° *If for n sufficiently large*

$$\alpha_n \geqslant 0 \wedge \alpha_n - \alpha_{n+1} \geqslant 0 \wedge \alpha_{n+1} - 2\alpha_n + \alpha_{n-1} \geqslant 0,$$

then for each $\theta \in (0, 2\pi)$

$$\sum_{k=0}^{n} e^{k\theta i} \alpha_k = \frac{e^{(n+1)\theta i}\alpha_n - T - \alpha_0}{e^{\theta i} - 1} + 0(\alpha_n - \alpha_{n+1}) \qquad (n \to +\infty),$$

where $T = \sum_{k=0}^{+\infty} e^{k\theta i}(\alpha_k - \alpha_{k+1})$.

2° *If in addition*

$$\lim_{n=+\infty} \alpha_n = 0,$$

then for any $\theta \in (0, 2\pi)$

$$\sum_{k=n}^{+\infty} e^{k\theta i} \alpha_k = \frac{1}{1 - e^{\theta i}} e^{n\theta i} \alpha_n + 0(\alpha_{n-1} - \alpha_n) \qquad (n \to +\infty).$$

Proof. 1° The condition of this case implies that the sequence (α_n) for n sufficiently large decreases and converges to a nonnegative finite number. This fact, as in previous cases, implies the (absolute) convergence of series $\sum_{k=0}^{+\infty} e^{(k+1)\theta i}(\alpha_k - \alpha_{k+1})$ and $\sum_{k=0}^{+\infty} t_k(\alpha_k - \alpha_{k+1})$ $(0 < \theta < 2\pi)$. In the same manner as in the proof of case 2° of Theorem 3, we get the equality

$$\left| \sum_{k=0}^{n} e^{k\theta i} \alpha_k - \frac{e^{(n+1)\theta i} - T - \alpha_0}{e^{\theta i} - 1} \right| =$$

$$\frac{1}{|e^{\theta i} - 1|} \left| t_n(\alpha_n - \alpha_{n+1}) - \sum_{k=n+1}^{+\infty} t_k(\alpha_{k-1} - 2\alpha_k + \alpha_{k+1}) \right| \qquad (0 < \theta < 2\pi;\ n \geqslant 0).$$

Further, with n large enough,

$$\left| \sum_{k=0}^{n} e^{k\theta i} \alpha_k - \frac{e^{(n+1)\theta i}\alpha_n - T - \alpha_0}{e^{\theta i} - 1} \right| \leqslant$$

$$\frac{1}{2\sin^2 \frac{\theta}{2}} \left\{ \alpha_n - \alpha_{n+1} + \sum_{k=n+1}^{+\infty} [(\alpha_{k-1} - \alpha_k - (\alpha_k - \alpha_{k+1}))] \right\} =$$

$$\frac{1}{\sin^2 \frac{\theta}{2}} (\alpha_n - \alpha_{n+1}) = 0(\alpha_n - \alpha_{n+1}) \qquad (n \to +\infty).$$

2° Under the supposition of this case, the series $\sum_{k=0}^{+\infty} e^{k\theta i}\alpha_k$ converges too, and so, by (9),

$$\left|\sum_{k=n}^{+\infty} e^{k\theta i}\alpha_k - \frac{1}{1-e^{\theta i}}e^{n\theta i}\alpha_n\right| =$$

$$\frac{1}{|e^{\theta i}-1|}\left|t_n(\alpha_n-\alpha_{n-1}) - \sum_{k=n+1}^{+\infty} t_k(\alpha_{k-1}-2\alpha_k+\alpha_{k+1})\right| \leqslant$$

$$\frac{1}{2\sin^2\frac{\theta}{2}}\left\{\alpha_{n-1}-\alpha_n+\sum_{k=n+1}^{+\infty}[\alpha_{k-1}-\alpha_k-(\alpha_k-\alpha_{k+1})]\right\} =$$

$$\frac{\alpha_{n-1}-\alpha_n}{\sin^2\frac{\theta}{2}} = 0(\alpha_n-\alpha_{n-1}) \qquad (n\to+\infty).$$

Corollary 2. *If all conditions of the assertion* 2° *of Theorem* 4 *are satisfied and*

$$\lim_{n\to+\infty}\frac{\alpha_{n+1}}{\alpha_n}=1,$$

then

$$\sum_{k=n}^{+\infty} e^{k\theta i}\alpha_k \sim \frac{1}{1-e^{\theta i}}e^{n\theta i}\alpha_n \qquad (n\to+\infty).$$

3. Some parts of the results in this paper, namely the confirmation of Hypothesis (i.e. Theorem 1), some assertions of Theorems 2—4 and Corollaries 1—2, generalize assertions 1° and 2° of Theorem 3 in [1] and Theorem 2 in [2]. The papers [1] and [2] were at the first place related-in the case when (a_n) is a sequence of complex numbers different from zero for n sufficently large, $S_n=\sum_{k=1}^{n} a_k$, $R_n=\sum_{k=n}^{+\infty} a_k$, $S=R_0$ and w complex number — to the question of relationship between the conditions

$$(\mathrm{I}_w) \qquad \lim_{n\to+\infty}\frac{S_n}{a_n}=w$$

and

$$(\mathrm{I}'_w) \qquad \lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\frac{w}{w-1}\wedge S=0 \text{ if series } \sum_{k=0}^{+\infty} a_k \text{ converges,}$$

and also between the conditions

$$(\mathrm{II}_w) \qquad \lim_{n\to+\infty}\frac{R_n}{a_n}=w$$

and

$$(\mathrm{II}''_w) \qquad \lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\frac{w-1}{w}.$$

One of the main results of the mentioned papers, Theorem 2 in [1], establishes *the validity of the equivalences*

$$(\mathrm{I_w}) \Leftrightarrow (\mathrm{I'_w}) \quad and \quad (\mathrm{II_w}) \Leftrightarrow (\mathrm{II'_w})$$

whenever $\mathrm{Re}(w) \neq \frac{1}{2}$. The mentioned Theorem 3 in [1] and Theorem 2 in [2] refer to the *singular case* $\mathrm{Re}(w) = \frac{1}{2}$. In view of the above said, it seems suitable to replace these two theorems by a unique and more complete statement, which partly reproduces and partly generalize their assertions. Accordingly to a suggestion of the author of [1] and [2], we shall formulate it as follows.

Theorem G. *For any complex* w *with the property* $\mathrm{Re}(w) = \frac{1}{2}$:

1° *condition* $(\mathrm{I'_w})$ *is necessary and not sufficient for* $(\mathrm{I_w})$, *and condition* $(\mathrm{II'_w})$ *is necessary and not sufficient for* $(\mathrm{II_w})$;

2° *equalities* $(\mathrm{I_w})$ *and* $(\mathrm{II_w})$ *are effectively realizable; especially, for any such* w:

2°.1 *if* $\theta \in (0, 2\pi)$ *is determined by*

$$\frac{w}{w-1} = e^{\theta i} \tag{15}$$

and the sequence (1) *of real numbers has the property*

$$\lim_{n \to +\infty} \alpha_n = +\infty \wedge$$

$$\wedge \left\{ \begin{array}{c} \lim\limits_{n \to +\infty} \dfrac{\alpha_{n+1} - \alpha_n}{\alpha_{n+2} - \alpha_{n+1}} = 1 \\ \vee \\ \lim\limits_{n \to +\infty} \dfrac{\alpha_{n+1}}{\alpha_n} = 1 \wedge (\alpha_{n+1} - \alpha_n \geqslant 0 \wedge \alpha_{n+1} - 2\alpha_n + \alpha_{n-1} \geqslant 0) \text{ for } n \text{ large enough,} \\ \vee \\ (\alpha_{n+1} - \alpha_n \geqslant 0 \wedge \alpha_{n+2} - 2\alpha_n + \alpha_{n-1} \geqslant 0) \text{ for } n \text{ large enough,} \end{array} \right.$$

then $(\mathrm{I_w})$ *holds for*

$$a_n = e^{n\theta i} \alpha_n \qquad (n \in N_0); \tag{16}$$

2°.2 *if* $\theta \in (0, 2\pi)$ *is determined by*

$$\frac{w-1}{w} = e^{\theta i}, \tag{17}$$

and the sequence (1) *of real numbers has the property*

$$\lim_{n\to+\infty} \alpha_n = 0 \wedge$$

$$\wedge \begin{cases} \lim_{n\to+\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1} - \alpha_{n+2}} = 1 \\ \vee \\ \lim_{n\to+\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1 \wedge (\alpha_{n+1} - \alpha_n \geqslant 0 \wedge \alpha_{n+1} - 2\alpha_n + \alpha_{n-1} \geqslant 0) \text{ for } n \text{ large enough,} \end{cases}$$

then (II_w) *holds in the case of the sequence* (16).

Let us remark that for each complex number w with the property $\mathrm{Re}(w) = \frac{1}{2}$, any of two equalities (15) and (17) determines uniquely the number $\theta \in (0, 2\pi)$, because, for each such number w, $|(w-1)w^{-1}| = 1$ and $(w-1)w^{-1} \neq 1$.

REFERENCES

[1] D. D. Adamović, *Sur la convergence des rapports de la somme pnrtielle au terme général et du reste au terme général d'une série réelle ou complexe*, Publ. de l'Ins. Math., Belgrade 1973, **15 (29)**, 5—20.

[2] D. D. Adamović, *Quelques compléments aux résultats du travail "Sur la convergence..."*, Publications de 'Inst. Math., Belgrade 1975, **20 (34)**, 9—27.

[3] D. D. Adamović, *Sur quelques propriétés des fonctions à croissance lente de Karamata* (I), Matematički vesnik **3 (18)**, 1966, sv. 2, 123—136.