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ON SOME CLASSES OF DIFFERENTIAL EQUATIONS

Bogdan Rzepecki

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In this paper we present results on the existence, uniqueness and continuous dependence on given functions of a solution for the Cauchy problem with retarded argument. These facts follow as a consequence of some applications of the "generalized metric space" (not every two points have necessarily finite distance) (see Luxemburg [8]).

Assumptions (i), (ii) and (iii), given below, are valid throughout this paper and will not be repeated in the formulations of particular theorems.

Suppose that:

(i) $I=[0, a]$, E is a Banach space with the norm $\|\cdot\|$;

(ii) the function $g: I\to(-\infty, \infty)$ is continuous and $g(t)\leqslant t$ for every $t\in I$, and let $m=\min\{g(t): t\in I\}$;

(iii) the function $\varphi:[m, 0]\to E$ is continuous, and the function $\tilde{\lambda}:[m, a]\to(-\infty, \infty)$ satisfies the following conditions: $\tilde{\lambda}(t)>0$ for every $t\neq 0$, $\lambda=\tilde{\lambda}|_I$ is a bounded function on I, and $\sup\left\{(\lambda(t))^{-1}\cdot\int_0^t\lambda(s)\,ds: 0<t\leqslant a\right\}<\infty$.

We shall consider the problem of finding the solution of the equation

$$(+)\qquad x'(t)=f(t, x(t), x(g(t)))$$

satisfying the initial condition

$$(++)\qquad x(t)=\varphi(t)\quad\text{for}\quad m\leqslant t\leqslant 0,$$

where x denotes an unknown function, g, φ and f from $I\times E\times E$ into E are given continuous functions.

1. O. Kooi [4] (comp. [8]) has proved the uniqueness of a solution of the Cauchy problem for an equation $x'=f(t, x)$ and the uniform convergence of sequence of successive approximations for this problem to this unique solution, when continuous bounded function f had satisfied the following conditions:

$$t\cdot|f(t,u)-f(t,v)|\leqslant K|u-v|,\quad t^{\beta}\cdot|f(t,u)-f(t,v)|\leqslant L|u-v|^{\alpha}$$

for $0<t\leqslant a$ and $-\infty<u, v<\infty$, where $K>0$, $L>0$, α, β are constants such that $0<\alpha<1$, $\beta<\alpha$, $K(1-\alpha)<1-\beta$. (For $\beta=0$ we obtain conditions for the uniqueness of M. A. Krasnoselskiî and I. G. Krein [5].)

We generalize the result of Kooi, applying the theorem on contraction in a generalized metric space [8].

A generalized metric space (X, d) is a pair composed of a non-empty set X and a distance function $d: X\times X\to[0, +\infty]$ satisfying the usual axioms for a metric space. If further: every d-Cauchy sequence in X is d-convergent (i. e., $\lim_{p,q\to\infty} d(x_p, x_q)=0$ for a sequence (x_n) in X, implies the existence of an element x_0 in X such that $\lim_{n\to\infty} d(x_n, x_0)=0$), then X is called a generalized complete metric space.

Our results are proved by the following theorem of the type of Banach fixed-point principle [6]:

Let (X, d) be a generalized complete metric space, let T_0 and T_n $(n=1, 2, \ldots)$ be mappings of X into itself such that $\lim_{n\to\infty} d(T_n x, T_0 x)=0$ for all x in X. Suppose, that there exist an element $z_0\in X$ and a constant k, $0\leqslant k<1$, such that $d(z_0, T_n z_0)<\infty$ $(n=1, 2, \ldots)$ and $d(T_n x, T_n y)\leqslant k\cdot d(x, y)$ $(n=1, 2, \ldots)$ for all $x, y\in X$ with $d(x, y)<\infty$. Then, the equation $T_m x=x$ $(m=0, 1, \ldots)$ has one and only one solution $u_m\in X$ such that $d(u_m, z_0)<\infty$ and $d(u_n, u_0)\to 0$ as $n\to\infty$. Moreover, every sequence of successive approximations $x_n^{(m)}=T_m x_{n-1}^{(m)}$ $(n=1, 2, \ldots)$, where $d(x_0^{(m)}, z_0)<\infty$, is d-convergent to this unique solution u_m.

2. For all continuous functions x, y from $[m, a]$ into E, which are identically equal to the function φ on an interval $[m, 0]$, we define the following distance function:

$$d(x, y)=\sup\{(\lambda(t))^{-1}\cdot\|x(t)-y(t)\|: 0<t\leqslant a\}.$$

We have $(\sup_{0<t\leqslant a}\lambda(t))^{-1}\cdot\sup_{m\leqslant t\leqslant a}\|x(t)-y(t)\|\leqslant d(x, y)$. This shows that d-convergence implies uniform convergence. Therefore, modifying the proof from [8], we can prove that every d-Cauchy sequence is d-convergent.

Let us denote:

by $\mathfrak{X}$ — the generalized complete metric space of all continuous functions from $[m, a]$ into E which are identically equal to the function φ on $[m, 0]$, with the distance function d;

by $C([m, a], E)$ — the space of all continuous function from $[m, a]$ into E, with the usual supremum norm $|||\cdot|||$;

by $\mathcal{F}$ — the set of continuous function f from $I \times E \times E$ into E satisfying the following conditions:

$$1^\circ \quad \left\| z_0(t) - \varphi(0) - \int_0^t f(s,\ z_0(s),\ z_0(g(s)))\, ds \right\| = 0(\lambda(t)) \quad \text{for every } 0 < t \leqslant a,$$

where $z_0 \in \mathfrak{X}$ is some function not depending on f,

2° there exist functions L_f, Q_f (which may depend on f) from I into $[0, +\infty]$ such that $\|f(t, u, v) - f(t, u_1, v_1)\| \leqslant L_f(t) \cdot \|u - u_1\| + Q_f(t) \cdot \|v - v_1\|$ for every $0 < t \leqslant a$ and u, v, u_1, v_1 in E.

In the sequel, we shall deal with the set $\mathcal{F}$ as an $\mathcal{L}^*$-space (see [7]) endowed with convergence: $\lim_{n \to \infty} f_n = f_0$ meaning

$$\sup\{(\lambda(t))^{-1} \cdot \|f_n(t, u, v) - f_0(t, u, v)\| :$$
$$: 0 < t \leqslant a,\ u \in \Omega_1,\ v \in \Omega_2\} \to 0 \text{ as } n \to \infty$$

for all compacts Ω_1, Ω_2 in E.

The equation $(+)$ with the condition $(++)$ is equivalent to the equation

$$(*) \qquad x(t) = \begin{cases} \varphi(t) & \text{for } m \leqslant t \leqslant 0, \\ \varphi(0) + \int_0^t f(s,\ x(s),\ x(g(s)))\, ds & \text{for } t \in I. \end{cases}$$

The successive approximations of the solution of (*) on I are defined by

$$v_{j+1}(t) = \begin{cases} \varphi(t) & \text{for } m \leqslant t \leqslant 0, \\ \varphi(0) + \int_0^t f(s,\ v_j(s),\ v_j(g(s)))\, ds & \text{for } t \in I \end{cases}$$
$$(j = 0, 1, \ldots),$$

where $f \in \mathcal{F}$ and v_0 is an arbitrary function in $\mathfrak{X}$ such that $d(v_0, z_0) < \infty$.

Let us put:

$$U_f(t) = \lambda(t) \cdot L_f(t), \quad V_f(t) = \tilde{\lambda}(g(t)) \cdot Q_f(t)$$

for $t \in I$. Now we shall prove

Theorem 1. *Suppose that for $f \in \mathcal{F}$ the functions U_f, V_f are integrable on the interval I, and*

$$k_f = \sup_{0 < t \leqslant a} \frac{1}{\lambda(t)} \int_0^t (U_f(s) + V_f(s))\, ds < 1.$$

Then, there exists the unique function $x_f \in \mathfrak{X}$ (given as the d-limit of successive approximations of the solution of (*)) satisfying the equation $(+)$ on I and such that $d(x_f, z_0) < \infty$.

Assume, moreover, that each U_f, $V_f (f \in \mathcal{F})$ is integrable on I, and $\sup\{k_f : f \in \mathcal{F}\} < 1$. *Then, the function $f \mapsto x_f$ maps continuously $\mathcal{F}$ into $\mathfrak{X}$.*

Proof. Let $f_m \in \mathcal{F}$ for $m = 0, 1, \ldots,$ and let $\lim_{n\to\infty} f_n = f_0$. We define the mappings $T_m (m = 0, 1, \ldots)$ on $\mathfrak{X}$ by the formula

$$(T_m x)(t) = \begin{cases} \varphi(t) \text{ for } m \leqslant t \leqslant 0, \\ \varphi(0) + \int_0^t f_m(s, x(s), x(g(s)))\,ds \text{ for } t \in I. \end{cases}$$

It is easy to verify that T_m maps $\mathfrak{X}$ into itself, and $d(z_0, T_n z_0) < \infty$ for each $n \geqslant 1$. We observe that if $x \in \mathfrak{X}$, then for $0 < t \leqslant a$ and $n \geqslant 1$

$$d(T_n x, T_0 x) \leqslant \sup_{0<t\leqslant a} \frac{1}{\lambda(t)} \int_0^t \lambda(s)\,ds.$$

$$\cdot \sup_{0<t\leqslant a} \frac{\|f_n(t, x(t), x(g(t))) - f_0(t, x(t), x(g(t)))\|}{\lambda(t)}$$

hence $d(T_n x, T_0 x) \to 0$ as $n \to \infty$.

Let us fix $0 < t \leqslant a$, $n \geqslant 1$ and x, y in $\mathfrak{X}$. We have

$$\|f_n(t, x(t), x(g(t))) - f_n(t, y(t), y(g(t)))\| \leqslant$$

$$\leqslant L_{f_n}(t)\|x(t) - y(t)\| + Q_{f_n}(t)\|x(g(t)) - y(g(t))\| \leqslant$$

$$\leqslant (U_{f_n}(t) + V_{f_n}(t)) \cdot d(x, y)$$

and hence, if $d(x, y) < \infty$, then $\|(T_n x)(t) - (T_n y)(t)\| \leqslant d(x, y) \cdot \int_0^t (U_{f_n}(s) + V_{f_n}(s))\,ds$. This implies, that $d(T_n x, T_n y) \leqslant k_{f_n} \cdot d(x, y)$ for $x, y \in \mathfrak{X}$ with $d(x, y) < \infty$.

Consequently, by the Banach fixed-point type theorem given in Sec 1, there exists the unique function $x_m \in \mathfrak{X}$ such that $(T_m x_m)(t) = x_m(t)$ $(m = 0, 1, \ldots)$ for $m \leqslant t \leqslant a$, and moreover $d(x_n, x_0) \to 0$ as $n \to \infty$. This completes the proof.

To the above result the following remarks should be added:

1. *Suppose that $|g(t)| \leqslant t$ for every $t \in I$, and f is a bounded continuous function from $I \times E \times E$ into E such that*

$$t \cdot \|f(t, u, v) - f(t, u_1, v_1)\| \leqslant K(\|u - u_1\| + \|v - v_1\|),$$

$$t^{\beta} \cdot \|f(t, u, v) - f(t, u_1, v_1)\| \leqslant L(\|u - u_1\|^{\alpha} + \|v - v_1\|^{\alpha})$$

for $0 < t \leqslant a$ and u, v, u_1, v_1 in E, where $K > 0$, $L > 0$, α, β are constants. Let us put: $\bar{\lambda}(t) = |t|^{pK}$ for $m \leqslant t \leqslant a$, $L_f(t) = Q_f(t) = t^{-1}K$ for $t \in I$, where p is some constant. If $0 < \alpha < 1$, $\beta < \alpha$, $2K(1-\alpha) < 1 - \beta$, $p > 2$ and *$pK(1-\alpha) < 1 - \beta$, then for f the assumptions of the first part of Theorem* 1 *are satisfied.*

Proof. Indeed, $\sup\limits_{0<t\leqslant a} (\lambda(t))^{-1}\cdot\int\limits_0^t \lambda(s)\,ds=(1+pK)^{-1}a$, $\int\limits_0^a U_f(t)\,dt=$ $=p^{-1}\cdot a^{pK}$, $\int\limits_0^a V_f(t)\,dt\leqslant K\cdot\int\limits_0^a t^{pK-1}\,dt=p^{-1}\cdot a^{pK}$ and $k_f\leqslant 2K\cdot\sup\limits_{0<t\leqslant a} t^{-pK}\int\limits_0^t\times$ $s^{pK-1}\,ds=2\cdot p^{-1}<1$. Modifying the proof from [8, p. 544] we can show, that there exists $z_f\in\mathfrak{X}$ such that $\left\|z_f(t)-\varphi(0)-\int\limits_0^t f(s,\ z_f(s),\ z_f(g(s)))\,ds\right\|=0\,(t^{pK})$ for $0<t\leqslant a$, which completes the proof.

2. *Suppose that* $|g(t)|\leqslant t$ *for every* $t\in I$, *and* f *is a continuous function from* $I\times E\times E$ *into* E *such that* $\|f(t,u,v)\|\leqslant M\cdot t^q$ *for* $(t,u,v)\in I\times E\times E$, $t\cdot\|f(t,u,v)-f(t,u_1,v_1)\|\leqslant N(\|u-u_1\|+\|v-v_1\|)$ *for* $0<t\leqslant a$ *and* u, v, u_1, v_1 *in* E, *where* $M>0$, $N>0$ *and* $q>-1$ *are constants. Let us put:* $\tilde{\lambda}(t)=$ $=|t|^{q+1}$ *for* $m\leqslant t\leqslant a$, $L_f(t)=Q_f(t)=N\cdot t^{-1}$ *for* $t\in I$. *If* $q>2N-1$, *then for* f *the assumptions of the first part of Theorem* 1 *are satisfied.*

Proof. We have: $\sup\limits_{0<t\leqslant a} (\lambda(t))^{-1}\int\limits_0^t \lambda(s)\,ds=(q+2)^{-1}\cdot a$, $\int\limits_0^a U_f(t)\,dt=$ $=(q+1)^{-1}\cdot N\cdot a^{q+1}$. Since $|g(t)|\leqslant t$ for $t\in I$, then $\int\limits_0^a V_f(t)\,dt\leqslant N\cdot\int\limits_0^a t^q\,dt=$ $=(q+1)^{-1}\cdot N\cdot a^{q+1}$. Hence $k_f\leqslant 2N\cdot\sup\limits_{0<t\leqslant a} t^{-(q+1)}\cdot\int\limits_0^t s^q\,ds=2N(q+1)^{-1}<1$.

Let us put

$$z(t)=\begin{cases}\varphi(t) & \text{for } m\leqslant t\leqslant 0,\\ \varphi(0)+(q+1)^{-1}\cdot M\cdot t^{q+1} & \text{for } t\in I.\end{cases}$$

Then, $z\in\mathfrak{X}$ and $\left\|z(t)-\varphi(0)-\int\limits_0^t f(s,\ z(s),\ z(g(s)))\,ds\right\|=0\,(t^{q+1})$ for $0<t\leqslant a$.

3. *Suppose that* f *is a continuous function from* $I\times E\times E$ *into* E *such that* $\|f(t,u,v)-f(t,u_1,v_1)\|\leqslant Q(\|u-u_1\|+\|v-v_1\|)$ *for* $t\in I$ *and* u, v, u_1, v_1 *in* E. *If* $C>2Q$ *and* $\tilde{\lambda}(t)=\exp(C\cdot t)$ *for* $m\leqslant t\leqslant a$, *then the assumptions of Theorem* 1 *are satisfied.*

Proof. Indeed, $\sup\limits_t(\lambda(t))^{-1}\int\limits_0^t\lambda(s)\,ds\leqslant C^{-1}$, $\int\limits_0^a U_f(s)\,ds<C^{-1}\cdot Q\cdot\exp$ $(C\cdot a)$, $\int\limits_0^a V_f(s)\,ds\leqslant Q\cdot\int\limits_0^a\exp(C\cdot s)\,ds$ and $k_f\leqslant 2Q\cdot\sup\limits_t\exp(-Ct)\int\limits_0^t\exp(C\cdot s)\,ds\leqslant$ $\leqslant 2Q\cdot C^{-1}<1$. This completes our proof.

From remark 3 it follows that our theorem is connected with Bielecki's method [1] of norm changing in the theory of differential equations. Using the above remarks we can obtain some corollaries from Theorem 1. In Sec. 3 we give a result of this sort in case 3.

3. Denote by $\mathscr{H}$ the set of all continuous functions f from $I\times E\times E$ into E such that $\|f(t, u, v)-f(t, u_1, v_1)\|\leqslant Q_f(\|u-u_1\|+\|v-v_1\|)$ for $t\in I$ and u, v, u_1, v_1 in E, where $Q_f\geqslant 0$ is a constant depending on f.

Assume that $C_0=\sup\{Q_f: f\in\mathscr{H}\}<\infty$, $C>2C_0$ and $\tilde{\lambda}$ is as in the remark 3. Let us put $|||x|||_C=\sup\{\exp(-C\cdot t)\cdot\|x(t)\|: m\leqslant t\leqslant a\}$ for $x\in\in C([m, a], E)$. The norm is equivalent to the original norm $|||\cdot|||$, and $d(x, y)= =|||x-y|||_C$ for functions x, y of $C([m, a], E)$ which are equal identically to the function φ on $[m, 0]$. From Theorem 1 and remark 3 we obtain the following corollary:

Let the set $\mathscr{H}$ be considered as an $\mathscr{L}^*$-space endowed with the almost uniform covergence. For an arbitrary $f\in\mathscr{H}$ there exists the unique function $x_f\in C([m, a], E)$ satisfying the equation $(+)$ on I and such that $x_f(t)=\varphi(t)$ for $m\leqslant t\leqslant 0$, Moreover, if $\sup\{Q_f: f\in\mathscr{H}\}<\infty$ then the function $f\mapsto x_f$ maps continuously $\mathscr{H}$ into $C([m, a], E)$.

In [2] there are proved some local theorems on the existence, uniqueness and continuous dependence on the given function of solution of the functional-differential equation in Euclidean space. The proofs of these results are based on a Nadler's theorem [10]. (The corresponding problem for the differential equation has been investigated in [10, p. 582].) We generalize the main results from [2]. Let us note that the remaining results from this paper can be analogously strengthened.

Denote by $\mathscr{H}_0$ the subset in $\mathscr{H}$ consisting of uniformly bounded functions. The set $\mathscr{H}_0$ will be considered as an $\mathscr{L}^*$-space endowed with the pointwise convergence on $I\times E\times E$.

Theorem 2. *For an arbitrary $f\in\mathscr{H}_0$ there exists the unique function $x_f\in C([m, a], E)$ (given as the uniform convergence limit of successive approximations of the solution of (*)) satisfying the equation $(+)$ on I and such that $x_f(t)=\varphi(t)$ for $m\leqslant t\leqslant 0$. Moreover, if $\sup\{Q_f: f\in\mathscr{H}\}<\infty$ then the function $f\mapsto x_f$ maps continuously $\mathscr{H}_0$ into $C([m, a], E)$.*

Proof. Assume that $f_m\in\mathscr{H}_0\,(m=0, 1, \ldots)$ and C, $\tilde{\lambda}$, $|||\cdot|||_C$ are as the proof of the above corollary, and $T_m\,(m=0, 1, \ldots)$ are the mappings defined in the proof of Theorem 1.

Let us fix $x\in C([m, a], E)$ such that $x(t)=\varphi(t)$ for $m\leqslant t\leqslant 0$. The Lebesgue bounded convergence theorem*) implies that $\lim\limits_{n\to\infty}\int\limits_0^t f_n(s, x(s),$

*) Since the limit fuction f_0 is continuous, so the integral can be understood as the Riemann integral [9] of the function defined on an interval I with values in the Banach space E.

$x(g(s)))\,ds = \int_0^t f_0(s, x(s), x(g(s)))\,ds$ for $t \in I$. Hence, $\lim_{n\to\infty} \|(T_n x)(t) - (T_0 x)\| = 0$ for $m \leqslant t \leqslant a$. This implies, by the equicontinuity of sequence $(T_n x)$ on the compact $[m, a]$, that the sequence $(T_n x)$ converges uniformly on $[m, a]$ to $T_0 x$ as $n\to\infty$. Finally, $d(T_n x, T_0 x)\to 0$ as $n\to\infty$. The application of fixed-point type theorem (given in Sec. 1) completes the proof.

4. The above theorems can be formulated for systems of differential and integral-differential equations. We have the following examples:

1°. Let m denote the space of all bounded sequences of real numbers with the norm $\|(x_1, x_2, \ldots)\|_\infty = \sum_1^\infty 2^{-i}|x_i|$. A function F from $I\times m\times m$ into $(-\infty, \infty)$ is continuous at the point $P = (\bar{t}, (\bar{x}_n), (\bar{y}_n)) \in I\times m\times m$ if and only if (comp. [11]) for every $\varepsilon>0$ there exist a number $\eta>0$ and natural numbers N', N'' such that $|F(t;\ x_1, x_2, \ldots;\ y_1, y_2, \ldots) - F(\bar{t};\ \bar{x}_1, \bar{x}_2, \ldots;\ \bar{y}_1, \bar{y}_2, \ldots)| < \varepsilon$, when $|t-\bar{t}|<\eta$, $|x_i - \bar{x}_i|<\eta$ for $i = 1, 2, \ldots, N'$ and $|y_j - \bar{y}_j|<\eta$ for $j = 1, 2, \ldots, N''$.

Given a non-negative constant C, let m_C be the set of real sequences bounded by C. Then, evidently, the convergence in m_C is equivalent to the coordinate-wise convergence.

Suppose, that g is a continuous function such that $|g(t)|\leqslant t$ for $t\in I$, and the functions $\varphi_n\ (n = 1, 2, \ldots)$ from $[m, 0]$ into $(-\infty, \infty)$ are continuous and uniformly bounded by A. Consider the infinite system of differential equations

$$x_n'(t) = F_n(t;\ x_1(t), x_2(t), \ldots;\ x_1(g(t)), x_2(g(t)), \ldots) \qquad (n = 1, 2, \ldots),$$

where $F_n\ (n = 1, 2, \ldots)$ are given on $I\times m\times m$ continuous functions uniformly bounded by B.

Let us put:

$$C = A + (a+1)B,$$

$$\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots) \text{ for } m\leqslant t\leqslant 0,$$

$$f(t, x, y) = (F_1(t, x, y), F_2(t, x, y), \ldots) \text{ for } (t, x, y)\in I\times m\times m.$$

Then, $\varphi: [m, 0]\to m_C$, $f: I\times m_C\times m_C\to m_C$ are continuous functions, and the above infinite system of differential equations with initial conditions

$$x_n(t) = \varphi_n(t)\ (n = 1, 2, \ldots) \text{ for } m\leqslant t\leqslant 0$$

is equivalent to the problem of finding the solution of $(+)$ satisfying the $(++)$ in m_C.

By means of the application of Theorem 1 and remark 1 one can prove that the following theorem holds:

Suppose that there exist constants α, β, K, L, *such that*

$$t\cdot|F_n(t, x, y)-F_n(t, \bar{x}, \bar{y})|\leqslant K(\|x-\bar{x}\|_\infty+\|y-\bar{y}\|_\infty),$$

$$t^\beta\cdot|F_n(t, x, y)-F_n(t, \bar{x}, \bar{y})\leqslant L(\|x-\bar{x}\|_\infty^\alpha+\|y-\bar{y}\|_\infty^\alpha)$$

for every $n\geqslant 1$, $t\in I$ *and* x, y, $\bar{x}$, $\bar{y}$ *in* m_C. *If* $K>0$, $L>0$, $0<\alpha<1$, $\beta<\alpha$ *and* $2K(1-\alpha)<1-\beta$, *then there exists the unique solution of the above initial--value problem; this solution consists of uniformly bounded continuous functions on the interval* $[m, a]$.

2°. Let $\mathbf{R}^n$ be an n-dimensional Euclidean space with the norm $|\,|$, Ω the compact subset of the finite dimensional space, and $C(\Omega, \mathbf{R}^n)$ the Banach space of continuous functions from Ω to $\mathbf{R}^n$. We consider the equation

$$\frac{\partial z(t, \tau)}{\partial t}=\int_\Omega G[t, \tau, \sigma, z(t, \sigma), z(g(t), \sigma)]\,d\sigma$$

regarded as an ordinary differential equation in the space $C(\Omega, \mathbf{R}^n)$.

Denote: $\varphi=(\varphi_1, \ldots, \varphi_n)$, $G=(G_1, \ldots, G_n)$.

Suppose that φ from $[m, 0]$ into $C(\Omega, \mathbf{R}^n)$ and G from $I\times\Omega\times\Omega\times\mathbf{R}^n\times\mathbf{R}^n$ into $\mathbf{R}^n$ are given continuous functions, and $|G_i(t, \tau, \sigma, x, y)-G_i(t, \tau, \sigma, \bar{x}, \bar{y})|\leqslant L(|x-\bar{x}|+|y-\bar{y}|)$ $(i=1, 2, \ldots, n)$ for every $t\in I$, $\tau\in\Omega$, $\sigma\in\Omega$ and x, y, $\bar{x}$, $\bar{y}$ in $\mathbf{R}^n$.

Then, using Theorem 1 and remark 3 we obtain the following theorem:

There exists one and only one system $(z_1, z_2, \ldots z_n)$ *consisting of continuous on* $[m, a]\times\Omega$ *functions* z_i $(i=1, 2, \ldots, n)$ *such that*

$$\frac{\partial z_i(t, \tau)}{\partial_t}=\int_\Omega G_i[t, \tau, \sigma, z_1(t, \sigma), \ldots, z_n(t, \sigma),$$
$$z_1(g(t), \sigma), \ldots, z_n(g(t), \sigma)]\,d\sigma$$

for $(t, \tau)\in I\times\Omega$, *and* $z_i(t, \tau)=\varphi_i(t)(\tau)$ *for* $(t, \tau)\in[m, 0]\times\Omega$.

5. The Darboux problem for hyperbolic differential equation $z_{xy}=f(x, y, z)$ is the two-dimensional analog of the Cauchy problem for an equation $z'=f(t, z)$. Therefore, the results presented in this paper can also be obtained for equation $z_{xy}=f(x, y, z, z_x, z_y)$.

Moreover les us denote that results similar to ours hold for the continuous functions satisfying the conditions of Kooi type [3]:

$$|f(t, u)|\leqslant M\cdot t^p,\quad t^s|f(t, u)-f(t, v)|\leqslant L|u-v|^r$$

for $0<t\leqslant a$ and $-\infty<u$, $v<\infty$, where $M>0$, $L>0$, $p>-1$, $r\geqslant 1$ and s are constants such that $r(1+p)-s=p$, $(2M)^{r-1}L<(1+p)^r$. We can prove this in the same way as above, using instead of Banach fixed-point type theorem (given in Sec. 1) the following theorem:

Let (X, d) *be a generalized complete metric space, let* T_0 *and* $T_n (n=1, 2, \ldots)$ *be mappings of* X *into itself such that* $\lim_{n\to\infty} d(T_n x, T_0 x)=0$ *for all* x *in* X. *Suppose that there exist constants* $\varepsilon>0$, $0\leqslant k<1$ *and an element* z_0 *in* X *such that* $d(z_0, T_n z_0)\leqslant\varepsilon$ *for* $n\geqslant 1$, *and* $d(T_n x\ T_n y)\leqslant k\cdot d(x, y)$ *for* $n\geqslant 1$ *and* $x, y\in X$ *with* $d(x, y)\leqslant\varepsilon$. *Then the equation* $T_m x=x (m=0, 1, \ldots)$ *has one and only one solution* $u_m\in X$ *such that there exists a finite sequence* $x_0=z_0, x_1, \ldots, x_k=u_m$ *with* $d(x_{i-1}, x_i)\leqslant\varepsilon$ *for* $1\leqslant i\leqslant k$, *and* $d(u_n, u_0)\to 0$ *as* $n\to\infty$.

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Institute of Mathematics A. Mickiewicz University,
Matejki 48/49, Poznań, POLAND

or:
Osiedle Boh. II Wojny Swiatowej 43/13
61—385 Poznań (POLAND)