PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 25 (39), 1979, pp. 153—156

FIXED POINT THEOREMS IN A UNIFORM SPACE

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(Received February 10, 1977)

In two recent papers ([1], [2]) Acharya proved some fixed point theorems for mappings defined on a uniform space. In this paper we establish some fixed point theorems for mappings satisfying a more general contractive type condition than most of those appearing in [1] or [2].

Let $(X, \mathcal{U})$ be a sequentially complete uniform space. A sequence $\{x_n\}$ is said to converge to a point x of X if, for each member U of $\mathcal{U}$, there exists a positive integer N such that $(x_n, x) \in U$ for all $n \geq N$. For any pseudometric p on X and any $r > 0$, let $V_{(p, r)} = \{(x, y) \mid x, y \in X \text{ and } p(x, y) < r\}$. Let $\mathcal{P}$ be the family of pseudometrics on X generating the uniformity $\mathcal{U}$ and let $\mathcal{V}$ denote the family of all sets of the form $\bigcap_{i=1}^{n} V_{(p_i, r_i)}$, where $p_i \in \mathcal{P}$, $r_i > 0$, $i = 1, 2, \ldots, n$ (the integer n is not fixed). The collection $\mathcal{V}$ forms a base for the uniformity $\mathcal{U}$. For other properties of the uniformity used in this paper the reader may consult [1] or [2].

Let $\alpha(x, y)$, $\beta(x, y)$, $\gamma(x, y)$, $\eta(x, y)$, $\zeta(x, y)$ be nonnegative functions of x and y satisfying

$$\sup_{x, y \in X} \{\alpha(x, y) + \beta(x, y) + \gamma(x, y) + \eta(x, y) + \zeta(x, y)\} = \lambda < 1.$$

Let T be a self-mapping of X satisfying: for any $V_i \in \mathcal{V}$, $i = 1, 2, \ldots, 5$, and $x, y \in X$.

$$(x, y) \in V_1,\ (x, Tx) \in V_2,\ (y, Ty) \in V_3,\ x, Ty) \in V_4,\ (y, Tx) \in V_5$$

implies

(1) $\quad (Tx, Ty) \in \alpha(x, y) V_1 \circ \beta(x, y) V_2 \circ \gamma(x, y) T_3 \circ \eta(x, y) T_4 \circ \zeta(x, y) V_5.$

We first establish the following

Theorem 1. *Let T: $X \to X$ which satisfies* (1). *Then T has a unique fixed point z and $\{T^n x\}$ converges to z for each $x \in X$.*

Proof. Let $x \in X$, V any member of $\mathcal{V}$. Pick p to be the Minkowski pseudometric corresponding to V. Write $p(x, y) = r_1$, $p(x, Tx) = r_2$, $p(y, Ty) = r_3$, $p(x, Ty) = r_4$, and $p(y, Tx) = r_5$. Fix $\varepsilon > 0$. Then $(x, y) \in (r_1 + \varepsilon) V_1$, $(x, Tx) \in (r_2 + \varepsilon) V_2$, $(y, Ty) \in (r_3 + \varepsilon) V_3$, $(x, Ty) \in (r_4 + \varepsilon) V_4$, and $(y, Tx) \in (r_5 + \varepsilon) V_5$.

From (1), with $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$, etc.,

$$(Tx, Ty) \in \alpha(r_1 + \varepsilon) V_1 \circ \beta(r_2 + \varepsilon) V_2 \circ \gamma(r_3 + \varepsilon) V_3 \ \ \eta(r_4 + \varepsilon) V_4 \circ \zeta(r_5 + \varepsilon) V_5,$$

which implies

$$p(Tx, Ty) < \alpha(r_1 + \varepsilon) + \beta(r_2 + \varepsilon) + \gamma(r_3 + \varepsilon) + \eta(r_4 + \varepsilon) + \zeta(r_5 + \varepsilon).$$

Since ε is arbitrary,

$$\begin{aligned} (2) \quad p(Tx, Ty) &\le \alpha p(x, y) + \beta p(x, Tx) + \gamma p(y, Ty) + \eta p(x, Ty) + \zeta p(y, Tx) \\ &\le \lambda \max\{p(x, y),\ p(x, Tx),\ p(y, Ty),\ p(x, Ty),\ p(y, Tx)\}. \end{aligned}$$

In (2), if one replaces the pseudometric p by a metric d, then one has the contractive definition of Ćirić [3]. It has been shown in [6] that (2) is one of the most general contractive definitions for metric spaces, if p is metric. With p a pseudometric (2) becomes one of the most general contractive definitions for uniform spaces.

Define the sequence $\{x_n\}$ by $x_0 = x$, $x_1 = Tx_0, \dots, x_{n+1} = Tx_n, \dots$. Let $O(x, n) = \{x, Tx, \dots, T^n x\}$ and define $\delta_p[O(x, n)] \ \max_{0 \le i < j \le n} p(T^i x, T^j x)$. Using the same argument as in [3], we obtain, for $m > n \ge 0$, $p(x_m, x_n) \le \lambda^n p(x, Tx)/(1 - \lambda)$. Therefore $\{x_n\}$ is Cauchy. Since X is complete, $\{x_n\}$ converges. Call the limit z. From (2),

$$\begin{aligned} p(z, Tz) &\le p(z, x_{n+1}) + p(x_{n+1}, Tz) \\ &\le p(z, x_{n+1}) + \lambda \max\{p(x_n, z),\ p(x_n, x_{n+1}),\ p(z, Tz), \\ &\qquad p(x_n, Tz),\ p(z, x_{n+1})\}. \end{aligned}$$

Taking the limit as $n \to \infty$ we obtain $p(z, Tz) \le \lambda p(z, Tz)$, which implies that $p(z, Tz) = 0$. Therefore $(z, Tz), \in V$ for every V in $\mathcal{V}$ and $z = Tz$.

For uniqueness, assume z and w are fixed points of T. Let $V \in \mathcal{V}$ and let p denote the corresponding pseudometric. From (2), $p(z, w) \le \lambda p(z, w)$, which implies $p(z, w) = 0$; i.e., $(z, w) \le V$. Since V arbitrary, $z = w$.

Corollary 1. *Let f: $X \to X$ such that f^q satisfies* (1) *for some fixed positive integer q. Then f has a unique fixed point w and $\{f^{qn} x\}$ converges to z for each $x \in X$.*

Proof. Use Theorem 1 with $T = f^q$ to to obtain the result that f^q has a unique fixed point z: i.e., $z = f^q(z)$. Therefore $f(z) = f^{q+1}(z) = f^q(f(z))$, and $f(z)$ is also a fixed point of f^q. Uniquenesss implies $z = f(z)$.

Theorem 2. *Let T_i: $X \to X$, $i = 1, 2, \dots$, be a sequence of mappings, each of which satisfies* (1) *for the same functions α, β, γ, η, ζ. If $\{T_n\}$ converges pointwise to a function T, then T has a unique point z and $\{z_n\}$ converges to z, where each z_n is the unique fixed point corresponding to T_n.*

Proof. Let $V\in\mathcal{V}$ and p be the corresponding pseudometric. From Theorem 1 each T_n possess a unique fixed point z_n. Using T_n in (2), and then taking the limit as $n\to\infty$ shows that T satisfies (2) and thus possesses a unique fixed point z. That $\{z_n\}$ converges to z follows from [5, Theorem 5].

Theorem 3. *Let $\{T_n\}$ be a sequence of mappings, with fixed points z_n, such that each T_n: $X\to X$ and $\{T_n\}$ converges uniformly to a mapping T: $X\to X$ which satisfies* (1). *Then T has a unique fixed point z and $z_n\to z$.*

Proof. That T has a unique fixed point follows from Theorem 1. It remains to show that $z_n\to z$.

Let $V\in\mathcal{V}$ with corresponding pseudometric p,

$$A=\{z_1, z_2, \dots\}, \quad p(z_n, z)=p(T_n z_n, Tz)\leq p(T_n z_n, Tz_n)+p(Tz_n, Tz).$$

Since $T_n\to T$ uniformly, there exists a positive integer N such that, for all $n\geq N$, $(T_n z_n, Tz_n)\in V$ for each $z_n\in A$. From (2), $p(Tz_n, Tz)\leq\lambda\max\{p(z_n, z), p(z_n, Tz_n)\}$. For each n such that the maximum equals $p(z_n, z)$ we obtain $p(z_n, z)\leq(1/(1-\lambda))\,p(z_n, Tz_n)$. For each n such that the maximum is equal to $p(z_n, Tz_n)$, we obtain $p(z_n, z)\leq(1+\lambda)\,p(z_n, Tz_n)$. In either case, $p(z_n, z)\leq (p(z_n, Tz_n)/(1-\lambda)$, so that $(z_n, z)\in(1/(1-\lambda))\,V$ for all $n\geq N$. Since V is arbitrary, $z_n\to z$.

Let $\alpha(x, y)$, $\beta(x, y)$, $\gamma(x, y)$, $\eta(x, y)$ be nonnegative functions of x and y satisfying

$$\sup_{x,\, y\in X}\{\alpha(x, y)+\beta(x, y)+\gamma(x, y)+2\eta(x, y)\}=\lambda<1.$$

Let T_1, T_2 be self-mappings of X satisfying: for any $V_i\in V$, $i=1, 2, \dots, 5$ and $x, y\in X$,

$$(x, y)\in V_1,\quad (x, T_1 x)\in V_2,\quad (y, T_2 y)\in V_3,\quad (x, T_2 y)\in V_4,\quad (y, T_1 x)\in V_5$$

implies

$$(3)\qquad (T_1 x, T_2 y)\in\alpha(x, y)\,V_1\circ\beta(x, y)\,V_2\circ\gamma(x, y)\,V_3\circ\eta(x, y)\,V_4\circ\eta(x, y)\,V_5.$$

Theorem 4. *Let T_1, T_2 be a pair of self-mappings of X satisfying* (3). *Then T_1 and T_2 have a unique common fixed point z.*

Proof. Let $x_0\in X$. Define $\{x_n\}$ by x_0, $x_1=T_1 x_0$, $x_2=T_2 x_1, \dots, x_{2n}= T_2 x_{2n-1}$, $x_{2n+1}=T_1 x_{2n}, \dots$. Let $V\in\mathcal{V}$ with p the corresponding pseudometric. Write $p(x, y)=r_1$, $p(x, Tx)=r_2$, $p(y, Ty)=r_3$, $p(x, Ty)=r_4$, $p(y, Tx)=r_5$. Fix $\varepsilon>0$. Then $(x, y)\in(r_1+\varepsilon)\,V_1$, $(x, Tx)\in(r_2+\varepsilon)\,V_2$, $(y, Ty)\in(r_3+\varepsilon)\,V_3$, $(x, Ty)\in(r_4+\varepsilon)\,V_4$, and $(y, Tx)\in(r_5+\varepsilon)\,V_5$. From (3), with $\alpha=\alpha(x, y)$, $\beta=\beta(x, y)$, etc.,

$$(T_1 x, T_2 y)\in\alpha(r_1+\varepsilon)\,V_1\circ\beta(r_2+\varepsilon)\,V_2\circ\gamma(r_3+\varepsilon)\,V_3\circ\eta(r_4+\varepsilon)\,V_4\circ\eta(r_5+\varepsilon)\,V_5,$$

which implies

$$p(T_1 x, T_2 y)<\alpha(r_1+\varepsilon)+\beta(r_2+\varepsilon)+\gamma(r_3+\varepsilon)+\eta(r_4+r_5+2\varepsilon).$$

Since ε is arbitrary,

(4) $$p(T_1 x, T_2 y) \leq \alpha p(x, y) + \beta p(x, Tx) + \gamma p(y, Ty) + \eta p(x, Ty) + \eta p(y, Tx) \leq \lambda \max\{p(x, y), p(x, Tx), p(y, Ty), [p(x, Ty) + p(y, Tx)]/2\}.$$

The result now follows from [4, Theorem 1]. The following corollary is easily obtained by combining the technique of the above theorem along with [4, Theorem 2].

Corollary 2. *Let T_1, T_2 be self mappings of X. Suppose there exists integers p and q such that T_1^p, T_2^q satisfy* (3). *Then T_1 and T_2 have a common unique fixed point z.*

Comments 1. Theorem 4 generalizes Theorems 3.2—2.5 of [1], and Corollary 2 generalizes Corollaries 3.2.1—3.5.1 of [1]. It should be noted that, in [1] the hypothesis of Theorem 3.3 and Corollary 3.3.1 should be amended to have $\alpha = \beta$ and $2\alpha < 1$. In Theorem 3.5 and Corollary 3.5.1 one must have $\alpha = \beta$. For, otherwise the indicated proofs fail.

2. Theorem 1 generalizes Theorems 3.1, 3.3 and 3.5 of [2].

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