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SOME RESULTS ABOUT APPROXIMATION OF SETS BY FINITE SETS IN NORMED SPACES

Lazar Pevac

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Let $S_n=\{y_1, y_2, \ldots, y_n\}$ be a set of n points in a normed space E and p a seminorm on E.

The *p-covering radius* of a set $M\subset E$ is the number

$$\operatorname{dist}_p(M, S_n)=\sup_{x\in M}\ \min_{1\leqslant i\leqslant n} p(x-y_i)$$

The set $\overline{S}_n=\{\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n\}$ is a *best n_p-net* for the set $M\subset E$ if

$$\operatorname{dist}_p(M, \overline{S}_n)=\inf_{S_n} \operatorname{dist}_p(M, S)$$

In the present paper we are concerned with the existence of such best n_p-nets.

Let us first prove some properties of a special kind of seminorms that will be used later.

Lemma. *If p is a seminorm on a conjugate space E and the set*

$$K_p=\{x\in E \mid p(x)\leqslant 1\}$$

is weakly closed, then*

1° $x_n \xrightarrow{\text{weak*}} x$ *implies* $\varliminf_{n\to\infty} p(x_n)\geqslant p(x)$

2° $x_n^i \xrightarrow{\text{weak*}} x^i$ *for* $1\leqslant i\leqslant m$ *implies*

$$\varliminf_{n\to\infty} \operatorname{dist}_p(M, \{x_n^1, x_n^2, \ldots, x_n^m\})\geqslant \operatorname{dist}_p(M, x^1, x^2, \ldots, x^n\})$$

Proof. 1°. Suppose, on the contrary, that there exists a $q>0$ such that

$$\varliminf_{n\to\infty} p(x_n)=p(x)-q$$

Then there exists a subsequence (x_{n_k}) such that

$$\varliminf_{n\to\infty} p(x_{n_k}) = p(x) - q$$

And so x_{n_k} for $k \geqslant k_0$ (where k_0 is large enough) lie in the set

$$K = \{y \in E \mid p(y) \leqslant p(x) - q/2\}$$

As K is weakly* closed we conclude that $x \in K$. This means that

$$p(x) \leqslant p(x) - q/2$$

which is impossible since $q > 0$.

Property 1° means that the seminorm p is sequentially weakly* lower semicontinuous.

Using it we shall prove 2°.

Let

$$\varliminf_{n\to\infty} \mathrm{dist}_p(M, \{x_n^1, \ldots, x_n^m\}) = r$$

and take subsequence $(x_{n_k}^i)$, $i = 1, 2, \ldots, m$, such that

$$\mathrm{dist}_p(M, \{x_{n_k}^1, \ldots, x_{n_k}^m\}) \leqslant r + 1/k$$

For an arbitrary $m' \in M$ the following holds

$$\min_{1\leqslant i\leqslant m} p(m' - x_{n_k}^i) \leqslant r + 1/k$$

In view of 1°, since, $m' - x_{n_k}^i$ converges weakly* to $m' - x^i$, we have

$$\min_{1\leqslant i\leqslant m} p(m' - x^i) \leqslant r$$

for arbitrary $m' \in M$. Thus

$$\mathrm{dist}_p(M, \{x^1, \ldots, x^m\}) = \sup_{m'\in M} \min_{1\leqslant i\leqslant m} p(m' - x^i) \leqslant r$$

which proves 2°.

Now we give a theorem which resolves the problem of existence of best n_p-nets in some particular cases.

Theorem 1. *Let p be a seminorm on a conjugate space E with the following properties*

(a) *the set $K_p = \{x \in E \mid p(x) \leqslant 1\}$ is weakly* closed*

(b) *there exists a $k > 0$ such that*

$$kp(x) \geqslant \inf_{y \in p^{-1}(0)} \| x + y \|$$

Then for every p-bounded set M ($\sup_{m\in M} p(m) < \infty$) there exists a best n_p-net.

Proof. We have by condition (a), that the subspace $p^{-1}(0)$ is weakly* closed in E since

$$p^{-1}(0)=\bigcap_{k=1}^{\infty}\{x\in E\mid p(x)\leqslant 1/k\}$$

Hence in the space F, whose dual is congruent to E, there exists a subspace Y on which all functionals from $p^{-1}(0)$ are anihilated. Moreover, it follows that

$$E/p^{-1}(0)\cong Y'. \tag{1}$$

Thus, $E/p^{-1}(0)$ is a conjugate space too. $E/p^{-1}(0)$ is normed by

$$\|\hat{x}\|=\inf_{y\in p^{-1}(0)}\|x+y\|$$

where

$$\hat{x}=x+p^{-1}(0).$$

We can define a new norm on $E/p^{-1}(0)$ by

$$\|\hat{x}\|_1=p(x)$$

The definition is correct, since $\hat{x}=\hat{y}$ implies $((x-y)\in p^{-1}(0)$ and

$$|p(x)-p(y)|\leqslant p(x-y)=0.$$

Hence $p(x)=p(y)$. The norm properties can also be proved easily.

We shall now find the best $n_{\|\ \|_1}$-net for $\hat{M}$ in $E/p^{-1}(0)$.

Let us first prove that the set $\hat{K}_p$ is weakly* closed. By virtue of the congruence (1) it is sufficient to find a $f\in Y$ which separates K_p and $w\notin K_p$.

Since K_p is absolutely convex and weakly* closed, we infer by the theorem Hahn-Banach, that there exists $f\in F$ such that $f(w)>1$ and $f(K_p)\leqslant 1$.

Let us show that $f\in Y$. Suppose, on the contrary, that there exists $x\in p^{-1}(0)$ such that $f(x)=q>0$. This would imply the existence of a number $t>0$ such that $f(tx)>1$, which is impossible, since tx must be contained in K_p. This proves that $f\in Y$.

Thus, it follows that the norm $\|\ \|_1$ satisfies the assumptions of the Lemma.

Let

$$\inf_{\hat{S}_n}\operatorname{dist}_{\|\ \|_1}(\hat{M},\hat{S}_n)=r$$

where $\hat{S}_n$ is a set of n points from $E/p^{-1}(0)$. Then there exists a sequence of sets $\hat{S}^i_n=\{\hat{y}^i_n,\dots,\hat{y}^1_n\}$ $(i\in N)$ such that

$$\operatorname{dist}_{\|\ \|_1}(M,\hat{S}^i_n)\leqslant r+1/i$$

Since M is p-bounded we have for $1\leqslant j\leqslant n$ and $i\in N$

$$\|y^i_j\|_1\leqslant\sup_{\hat{x}\in\hat{M}} p\,\|x\|_1+r+1<L. \tag{2}$$

Inequality (2) and condition (b) ensure that the sets $\hat{S}_n^i$ are contained in the sphere $\hat{S}(\hat{0}, kL)$. Since this sphere is weakly* compact, it can be found a subsequence $(\hat{S}_n^{i_k})$ such that

$$\hat{y}_j^{i_k} \xrightarrow{\text{weak*}} \hat{y}_j \qquad (1 \leqslant j \leqslant n) \tag{3}$$

This can be done in the following way. First we extract a weakly* convergent subsequence $(\hat{y}_1^{i_m})$ from the sequence $(\hat{y}_1^i)$ and then again we extract a weakly* convergent subsequence $(\hat{y}_2^{i_{m_j}})$ from the sequence $(\hat{y}_2^{i_m})$, continuing this procedure to the last sequence $(\hat{y}_n^i)$). The required subsequence of $(\hat{S}_n^i)$ will have the indices of the last subsequence of $(\hat{y}_n^i)$. Applying the Lemma to the subsequence $(\hat{S}_n^{i_k})$ which satisfies (3) we get

$$\operatorname{dist}_{\|\ \|_1}(\hat{M}, \{\hat{y}_1, \dots, \hat{y}_n\}) \leqslant r.$$

Hence, $\{\hat{y}_1, \dots, \hat{y}_n\}$ is the best $n_{\|\ \|_1}$-net for $\hat{M}$.

It remains only to return to the space E. We shall prove that the set $\{y_1, \dots, y_n\}$ is the best n_p-net for M.

If it were not, there would exist a set $\{z_1, \dots, z_n\}$ which would approximate M better than $\{y_1, \dots, y_n\}$. And since $p(x) = \|\hat{x}\|_1$, it is seen easily that the set $\{\hat{z}_1, \dots, \hat{z}_n\}$ would approximate $\hat{M}$ better than the one found before, which is impossible.

This completes the proof of the theorem.

We shall now try to modify the hypotesis in Theorem 1 and get the same conclusion.

Theorem 2. *Let E be a normed space and p a continuous seminorm such that*

(a) *$E/p^{-1}(0)$ is a conjugate space*

(b) *same is an Theorem* 1.

Then for every p-bounded set $M \subset E$ there exists a best n_p-net in E.

Proof. The norm $\|\ \|_1$ is continuous since p is continuous. Hence, there exists a $m > 0$ such that

$$\|\hat{x}\| \geqslant m \|\hat{x}\|_1$$

Using condition (b) we conclude that the norms $\|\ \|$ and $\|\ \|_1$ are equivalent. Because of this the space $E/p^{-1}(0)$ endowed with the topology induced by the norm $\|\ \|_1$ is a conjugate space. By virtue of Theorem 1. there exists a best $n_{\|\ \|_1}$-net $\{\hat{y}_1, \dots, \hat{y}_n\}$ for the set $\hat{M}$. Like in the preceding theorem it is easily seen that the set $\{y_1, \dots, y_n\}$ forms the best n_p-net for M.

Finally, we shall state two corollaries of Theorem 1 and 2. Let $n = 1$ and

$$p_H(x) = \inf_{y \in H} \|x + y\|$$

where H is a subspace of F.

C o r o l l a r y 1. *If F is a conjugate space and H is a subspace of E such that* $\overline{H}$ *is weakly* closed and M is a* p_H*-bounded set in E, then there exists a flat* x_0+H *which approximates M better than all the flats* $x+H$*, where* $x\in E$*.*

Obviously, the corollary is valid if H is a space of finite dimension. If E is a reflexive space, the conclusion is valid for every subspace H.

C o r o l l a r y 2. *If E is a normed space and H is a subspace of E such that* $E/\overline{H}$ *is a conjngate space (particularly H is a finite codimension) and M is a* p_H*-bounded set in E, then there exists a flat* x_0+H *which approximates M better than all the flats* $x+H$*, where* $x\in E$*.*

The expression "x_0+H approximates M better than all the flats $x+H$" means

$$\mathrm{dist}_{\|\ \|}(M, x_0+H)\leqslant \mathrm{dist}_{\|\ \|}(M, x+H)$$

where $\|\ \|$ is the norm on E.

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