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PRODUCT — CONCIRCULAR CURVATURE TENSOR

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1. Introduction.

In the paper [1] is investigated conformal transformation of a n-dimensional on Riemannian space V_n, given by

(1) $$\bar{g}_{ji} = \rho^2 g_{ji} \qquad i, j, = 1, 2, \ldots, n$$

which transforms any geodesic circle into a geodesic circle. In the same paper is proved that for such, so-called concircular, transformation condition

(2) $$\rho_{ji} = \Phi g_{ji}$$

must be satisfied, where ρ and φ are scalars, and

$$\rho_{ji} = \nabla_j \rho_i - \rho_j \rho_i + \frac{1}{2} g^{mn} \rho_m \rho_n g_{ji}.$$

$$\rho_i = \frac{\partial (\log \rho)}{\partial u^i}$$

∇ denotes the operator of covariant differentiation.

An invariant of the concircular transformation (1), (2) is the tensor

(3) $$Z_{kji}{}^h = R_{kji}{}^h + \frac{R}{n(n-1)} (\delta_j^h g_{ki} - \delta_k^h g_{ji})$$

where δ_j^i is the Kronecker symbol, R_{kji} Rieman-Christoffel curvature tensor and R curvature tensor.

$Z_{kji}{}^h$ is called *concircular curvature tensor.*

2. Concircular infinitesimal transformation.

Let us give conformal infinitesimal transformation

$$\text{(4)} \qquad \underset{v}{£}\, g_{ji} = 2\,\theta\, g_{ji} \;[1]$$

θ — scalar

of a Riemannian space V_n. From (4) we have

$$\text{(5)} \qquad \underset{v}{£} \left\{ \begin{matrix} \tilde{k} \\ ji \end{matrix} \right\} = \theta_j\, \delta_i^k + \theta_i\, \delta_j^k - \theta^k g_{ji} \;[2]$$

where $\theta_i = \dfrac{\partial\, \theta}{\partial\, u^i}$, and $\theta^i = g^{ia}\, \theta_a$ and

$$\text{(6)} \qquad \left\{ \begin{matrix} \tilde{k} \\ ji \end{matrix} \right\} = \left\{ \begin{matrix} k \\ ji \end{matrix} \right\} + (\theta_j\, dt)\, \delta_i^k + (\theta_i \cdot dt)\, \delta_j^k - (\theta^k \cdot dt)\, g_{ji} \;[3]$$

where $\left\{ \begin{matrix} \tilde{k} \\ ji \end{matrix} \right\}$ is transform of $\left\{ \begin{matrix} k \\ ji \end{matrix} \right\}$ with respect to transformation (4).

But from (1) we get

$$\text{(7)} \qquad \overline{\left\{ \begin{matrix} k \\ ji \end{matrix} \right\}} = \left\{ \begin{matrix} k \\ ji \end{matrix} \right\} + \rho_j\, \delta_i^k + \rho_i\, \delta_j^k - \rho^k\, g_{ji}$$

$$\rho_j = \frac{\partial\, (\log \rho)}{\partial\, u^j}, \quad \rho^i = g^{ia}\, \rho_a.$$

Comparing (6) and (7) we see that ρ_i, in the transformation (1), (2) is corresponding to $\theta_i\, dt$ in the transformation (4).

Now we get for the condition (2) the following form

$$\nabla_j\, (\theta_i\, dt) - (\theta_j\, dt)\, (\theta_i\, dt) = \sigma_1\, g_{ji} \qquad \sigma_1 \text{ — scalar}$$

from which

$$\text{(8)} \qquad \nabla_j\, \theta_i = \sigma\, g_{ji} \qquad \sigma = \frac{\sigma}{dt}$$

up to infinitesimals of first order with respect to dt.

By a straightforward calculation we get

$$\underset{v}{£}\, Z_{kji}{}^h = 0$$

for transformation (4), (8).

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[1] See [3] p. 31.
$\underset{v}{£}$ denotes the operator of Lie derivation with respect to v^i.

[2] See [2] p. 16.

[3] See [3] p. 4.

Definition: A transformation (4) which satisfies condition (8) is called *concircular infinitesimal transformation.*

3. Product-concircular infinitesimal transformation.

We shall consider, so-called, product-concircular infinitesimal transformation and find an invariant of it.

Let us give an n-dimensional locally product Riemannian space $V_n = V_p \times V_q$. Then, by definition[4)], there exists a system of coordinate neighbourhoods $\{U_\alpha\}$ such that in each U_α the line element is given by the form

$$ds^2 = g_{ba}(u^c)\,du^b\,du^a + g_{yx}(u^z)\,du^y\,du^x \; {}^{5)}$$

and in $U_\alpha \cap U_\beta$ the coordinate trasformation $(u^a, u^x) \to (u^{a'}, u^{x'})$ is given by the form

$$u^{a'} = u^{a'}(u^a), \qquad u^{x'} = u^{x'}(u^x)$$

Such coordinate system will be called a separating coordinate system.

The existence of a separating coordinate system is equivalent to the existence of a tensor field φ_i^h $(\varphi_i^h \neq \delta_i^h)$ with following properties:

$$\varphi_i^r \varphi_r^h = \delta_i^h$$

$$g_{ri}\varphi_j^r = g_{jr}\varphi_i^r = \varphi_{ji}$$

$$\nabla_j \varphi_i^h = 0$$

Matrices (g_{ji}), (φ_j^h) and (φ_{ji}) in the separating coordinate system are given by the form

$$(g_{ji}) = \begin{pmatrix} g_{ba} & 0 \\ 0 & g_{yx} \end{pmatrix}$$

$$(\varphi_j^h) = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_y^x \end{pmatrix}$$

(9) $$(\varphi_{ji}) = \begin{pmatrix} g_{ba} & 0 \\ 0 & -g_{yx} \end{pmatrix}$$

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4) See [2] p. 213.

5) The indices a, b, c, d, e run over the range $1, 2, \dots, p$, the indices t, x, y, z over the range $p-1, p+2, \dots, p+q=n$, and the indices h, i, j, k, l, r over the range $1, 2, \dots, p, p+1, \dots, n$. We denote by

$$g_{ba}(u^c) = g_{ba}(u^1, \dots, u^p) \text{ and } g_{yx}(u^z) = g_{yx}(u^{p+1}, \dots, u^n).$$

Let us apply transformation (4), (8) to each of the subspaces V_p and V_q, of a space V_n. We will get in V_p

$$\underset{v}{\pounds}\, g_{ba} = 2\lambda g_{ba} \qquad \nabla_b \lambda_a = \psi_1 g_{ba} \tag{10}$$

where $\lambda = \lambda(u^a)$ and $\psi_1 = \psi_1(u^a)$ are scalars, $\lambda_a = \frac{\partial \lambda}{\partial u^a}$,

and in V_q

$$\underset{v}{\pounds}\, g_{yx} = 2\mu g_{yx} \qquad \nabla_y \mu_x = \psi_2 g_{yx} \tag{11}$$

where $\mu = \mu(u^x)$ and $\psi_2 = \psi_2(u^x)$ are scalars, $\mu_x = \frac{\partial \mu}{\partial u^x}$.

From (9), (10), (11), we have the following transformation on V_n

$$\underset{v}{\pounds}\, g_{ji} = 2(\rho g_{ji} + \sigma \varphi_{ji}) \tag{12}$$

where $\rho = \rho(u^i)$ and $\sigma = \sigma(u^i)$ are scalars and satisfy the conditions

$$\rho = \frac{1}{2}(\lambda + \mu)$$

$$\sigma = \frac{1}{2}(\lambda - \mu)$$

$$\rho_i = \varphi_i^r \sigma_r \tag{13}$$

$$\nabla_j \rho_i = \frac{1}{2}(\theta_1 g_{ji} + \theta_2 \varphi_{ji}) \tag{14}$$

$$\nabla_j \sigma_i = \frac{1}{2}(\theta_2 g_{ji} + \theta_1 \varphi_{ji}) \tag{15}$$

$$\nabla_j \rho^h = \frac{1}{2}(\theta_1 \delta_j^h + \theta_2 \varphi_j^h) \tag{16}$$

$$\nabla_j \sigma^h = \frac{1}{2}(\theta_2 \delta_j^h + \theta_1 \varphi_j^h) \tag{17}$$

$$x = \nabla_r \rho^r = \frac{1}{2}(n\theta_1 + \varphi\theta_2) \tag{18}$$

$$y = \nabla_r \sigma^r = \frac{1}{2}(\varphi\theta_1 + n\theta_2) \tag{19}$$

where

$$\theta_1 = \frac{1}{2}(\psi_1 + \psi_2), \quad \theta_2 = \frac{1}{2}(\psi_1 - \psi_2), \quad n = \delta_r^r = p + q, \quad \varphi = \varphi_r^r = p - q.$$

D e f i n i t i o n: A transformation (12), (13), (14) of n-dimensional locally product Riemannian space V_n is called *product-concircular infinitesimal transformation.*

From above, by straight-forward calculation we obtain

$$\underset{v}{\pounds}\, g^{ji} = -2\,(\rho\, g^{ji} + \sigma\varphi^{ji}) \qquad \varphi^{ji} = \varphi_r^{j}\, g^{ri}$$

$$\underset{v}{\pounds}\, \varphi_{ji} = 2\,(\sigma\, g_{ji} + \rho\,\varphi_{ji})$$

$$\underset{v}{\pounds}\, \varphi^{ji} = -2\,(\sigma\, g^{ji} + \rho\varphi^{ji})$$

$$\underset{v}{\pounds}\, \left\{ {h \atop ji} \right\} = \rho_j\, \delta_i^h + \rho_i\, \delta_j^h - \rho^h g_{ji} + \sigma_j\, \varphi_i^h + \sigma_i\, \varphi_j^h - \sigma^h \varphi_{ji}$$

(20) $$\underset{v}{\pounds}\, R_{kji}{}^h = \delta_j^h\, \nabla_k\, \rho_i - \delta_k^h\, \nabla_j\, \rho_i + g_{ki}\, \nabla_j\, \rho^h - g_{ji}\, \nabla_k\, \rho^h + {}$$
$$+ \varphi_j^h\, \nabla_k\, \sigma_i - \varphi_k^h\, \nabla_j\, \sigma_i + \varphi_{ki}\, \nabla_j\, \sigma^h - \varphi_{ji}\, \nabla_k\, \sigma^h$$

$$\underset{v}{\pounds}\, R_{ji} = -(n-4)\, \nabla_j\, \rho_i - \varphi\, \nabla_j\, \sigma_i - x \cdot g_{ji} - y \cdot \varphi_{ji}$$

$$\underset{v}{\pounds}\, R_j^h = -(n-4)\nabla_j\, \rho^h - \varphi \cdot \nabla_j\, \sigma^h - x\, \delta_j^h - y \cdot \varphi_j^h -$$
$$-2\,(\rho\, R_j^h + \sigma\, R_j^{*h})$$

(21) $$\underset{v}{\pounds}\, R = -2\,(n-2)\,x - 2\,\varphi\, y - 2\,\rho\, R - 2\,\sigma\, R^*$$

$$\underset{v}{\pounds}\, \overset{*}{R}_{ji} = -\varphi \cdot \nabla_j\, \rho_i - (n-4)\, \nabla_j\, \sigma_i - y \cdot g_{ji} - x \cdot \varphi_{ji}$$

$$\underset{v}{\pounds}\, R_j^{*h} = -\varphi \cdot \nabla_j \cdot \rho^h - (n-4)\, \nabla_j\, \sigma^h - y \cdot \delta_j^h - x \cdot \varphi_j^h -$$
$$-2\,(\sigma\, R_j^h + \rho\, R_j^{*h})$$

(22) $$\underset{v}{\pounds}\, R^* = -2\,\varphi\, x - 2\,(n-2)\,y - 2\,\sigma\, R - 2\,\rho\, R^*$$

$$\underset{v}{\pounds}\, r_{kji}{}^h = 2\,(\rho\, r_{kji}{}^h + \sigma\, \overset{*}{r}_{kji}{}^h)$$

$$\pounds\, \overset{*}{r}_{kji}{}^h = 2\,(\sigma\, r_{kji}{}^h - \rho\, \overset{*}{r}_{kji}{}^h)$$

where

$$R_{kji}{}^h = \frac{\partial}{\partial u^k} \left\{ {h \atop ji} \right\} - \frac{\partial}{\partial u^j} \left\{ {h \atop ki} \right\} + \left\{ {h \atop kl} \right\} \left\{ {l \atop ji} \right\} - \left\{ {h \atop jl} \right\} \left\{ {l \atop ki} \right\}$$

$$R_{ji} = R_{rji}{}^r \qquad \overset{*}{R}_{ji} = R_{jr} \cdot \varphi_i^r$$

$$R_j^h = R_{jr}\, g^{rh} \qquad R_j^{*h} = R_{jr} \cdot \varphi^{rh} = R_j^r \cdot \varphi_r^h$$

$$R = R_r^r = R_{ji} g^{ji} \qquad R^* = R_r^{*r} = R_{ji} \varphi^{ji}$$

$$r_{kji}{}^h = \delta_j^h g_{ki} - \delta_k^h g_{ji} + \varphi_j^h{}_{ki} - \varphi_k^h \varphi_{ji}$$

$$\overset{*}{r}_{kji}{}^h = \varphi_j^h g_{ki} - \varphi_k^h g_{ji} + \delta_j^h \varphi_{ki} - \delta_k^h \varphi_{ji} = r_{kji}{}^r \cdot \varphi_r^k$$

Substituting (14), (15), (16), (17), (18), (19) in (20), (21), (22) we obtain

(20′) $$\underset{v}{£} R_{kji}{}^h = \theta_1 r_{kji}{}^h + \theta_2 \overset{*}{r}_{kji}{}^h$$

(21′) $$\underset{v}{£} R = -\theta_1 [n(n-2) + \varphi^2] - \theta_2 \cdot 2\varphi (n-1) 1 - 2(\rho R + \sigma R^*)$$

(22′) $$\underset{v}{£} R^* = -\theta_1 \cdot 2\varphi (n-1) - \theta_2 [n(n-2) + \varphi^2] - 2(\rho R + \sigma R^*)$$

and from this

$$\underset{v}{£} E_{kji}{}^h = 0$$

where

(23) $$E_{kji}{}^h = R_{kji}{}^h + (\alpha R + \beta R^*) r_{kji}{}^h + (\beta R + \alpha R^*) \overset{*}{r}_{kji}{}^h$$

$$\alpha = \frac{n(n-2) + \varphi^2}{[n(n-2) + \varphi^2]^2 - 4\varphi^2 (n-1)^2}$$

$$\beta = \frac{-2\varphi (n-1)}{[n(n-2) + \varphi^2]^2 - 4\varphi^2 (n-1)^2}$$

Tensor (23) is invariant under the transformation (12), (13), (14), and has following properties:

a) $E_{kji}{}^h$ is pure in all its indices, that is, all its components are identicaly zero except $E_{cba}{}^d$ and $E_{zyx}{}^t$. Moreover, $E_{cba}{}^d$ are functions of u^a only, and $E_{zyx}{}^t$ are functions of u^x only. The above statements follow from the corresponding properties of tensors $R_{kji}{}^h$ and $r_{kji}{}^h$.

b) 1° $E_{cba}{}^d = Z_{cba}{}^d$ 2° $E_{zyx}{}^t = Z_{zyx}{}^t$

1° Indeed,

$$E_{cba}{}^d = R_{cba}{}^d + (\alpha R + \beta R^*) r_{cba}{}^d + (\beta R + \alpha R^*) \overset{*}{r}_{cba}{}^d =$$

$$= R_{cba}{}^d + (\alpha R_{ji} g^{ji} + \beta R_{ji} \varphi^{ji}) r_{cba}{}^d +$$

$$+ (\beta R_{ji} g^{ji} + \alpha R_{ji} \varphi^{ji}) \overset{*}{r}_{cca}{}^d$$

Since

$$R_{ji} g^{ji} = R_{ba} g^{ba} + R_{yx} g^{yx} = \overset{(p)}{R} + \overset{(q)}{R}$$

$$R_{ji} \varphi^{ji} = R_{ba} \varphi^{ba} + R_{yx} \varphi^{yx} = \overset{(p)}{R} - \overset{(q)}{R}$$

where $\overset{(p)}{R}$ and $\overset{(q)}{R}$ are curvature scalars of the spaces V_p and V_q respectively and

$$r_{cba}{}^{d} = \overset{*}{r}_{cba}{}^{d} = 2\,(\delta_b^d g_{ca} - \delta_c^d g_{ba})$$

$$n = p + q$$

$$\varphi = p - q$$

we obtain finally

$$E_{cba}{}^{d} = R_{cba}{}^{d} + \frac{\overset{(p)}{R}}{p\,(p-1)}\,(\delta_b^d g_{ca} - \delta_c^d g_{ba}) = Z_{cba}{}^{d}$$

The proof of 2° is similar.

Since the projections of the tensor $E_{kji}{}^{h}$ into spaces V_p and V_q are concircular curvature tensors /b)/, tensor $E_{kji}{}^{h}$ we call the *product-concircular curvature tensor.*

It is interesting that the same tensor is obtained in the paper [5] in totaly different way.

BIBLIOGRAPHY

[1] Yano, K., *Concircular geometry I. Concircular transformations*, Mathematical Institut, Tokyo (1940)

[2] Yano, K., *Differential geometry on complex and almost complex manifolds*, Oxford 1965

[3] Yano, K., *The theory of Lie derivatives and its applications*, Amsterdam

[4] Tachibana, S., *Some theorems on locally product Riemannian spaces*, Ochanomizy University, Tokyo (1959)

[5] Prvanonić M., *On two tensors in a locally decomposable Riemannian spaces*, Zbornik radova Prirodno-matematičkog fakulteta, № 6, 1976. Novi Sad.