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ON A THEOREM CONCERNING UNIFORM INTEGRABILITY

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In this paper, we reprove a theorem of La Vallée Poussin [3, Theorem T 22, p. 19] which gives necessary and sufficient conditions for a family of integrable random variables to be uniformly integrable. Our proof is based on a characterization of uniform integrability given in [1, Theorem 4.2, p. 401] and a rearrangement theorem obtained in [2, Theorem 2.3, p. 1330]. Consequently, our result shows that the La Vallée Poussin theorem can be established for a family of integrable random variables by proving it only for a single random variable.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by $M(\Omega, \mathcal{F}, P)$ the collection of all real valued finite random variables on $(\Omega, \mathcal{F}, P)$. Let $(\Omega', \mathcal{F}', P')$ be another probability space which is not necessarily distinct from $(\Omega, \mathcal{F}, P)$. If $X, Y \in M(\Omega, \mathcal{F}, P) \cup M(\Omega', \mathcal{F}', P')$ are any two random variables with integrable positive parts, then we write $X \ll Y$ whenever

$$E[(X-t)^+] \leq E[(Y-t)^+] \tag{1}$$

for all $t \in R$.

Lemma. *Let $X \in M(\Omega, \mathcal{F}, P)$ be any nonnegative random variable which is not essentially bounded. If $0<p<1$, define a function $\Phi: R^+ \to R$ by*

$$\Phi(t) = \int_0^t \{E[(X-s)^+]\}^{-p} \, ds \tag{2}$$

for all $t \geq 0$. Then Φ is a nonnegative increasing convex function such that

$$\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty, \tag{3}$$

and

$$E[\Phi(X)] = \frac{1}{1-p} (E[X])^{1-p} \tag{4}$$

in the sense that if either side exists so does the other and then both are equal.

P r o o f. It is not hard to see that the function $s \to \{E[(X-s)^+]\}^{-p}$, where $s \geq 0$, is nonnegative increasing (convex) and that it tends to infinity with s since X is not essentially bounded. From this, it then follows easily that Φ is nonnegative, increasing and convex such that (3) holds.

It remains to prove (4). Assume $X \in L^1(\Omega, \mathcal{F}, P)$. Let $D_X : R \to [0, 1]$ be the function defined by $D_X(t) = P[\{\omega : X(\omega) > t\}]$, $t \in R$. Then it can be proved [2, Corollary 1.2, p. 1323] that

$$E[(X-s)^+] = \int_s^\infty D_X(t)\,dt, \qquad s \geq 0. \tag{5}$$

Thus, using [2, Proposition 1.1, p. 1323], we have $E[\Phi(X)] = \int_0^\infty D_X(t)\,d\Phi(t) =$

$$\int_0^\infty D_X(t)\left(\int_t^\infty D_X(u)\,du\right)^{-p} dt = \int_0^\infty \left(\int_0^\infty D_X(u)\,du - \int_0^t D_X(u)\,du\right)^{-p} d\left(\int_0^t D_X(u)\,du\right) =$$

$$\left[\frac{-1}{1-p}\left(E[X] - \int_0^t D_X(u)\,du\right)^{1-p}\right]_0^\infty = \frac{1}{1-p}(E[X])^{1-p}, \text{ since } E[X] = \int_0^\infty D_X(u)\,du$$

by (5). On the other hand, if $\Phi(X) \in L^1(\Omega, \mathcal{F}, P)$, then it follows as in the first part of the proof of [3, Theorem T22, p. 19] that X is integrable.

T h e o r e m (La Vallée Poussin [3, Theorem T22, p. 19]). *Let $\mathcal{H}$ be a subset of $M(\Omega, \mathcal{F}, P)$. Then $\mathcal{H}$ is uniformly integrable if and only if there exists a nonnegative increasing convex function $\Phi : R^+ \to R$ such that both* (3) *and*

$$\sup_{X \in \mathcal{H}} E[\Phi(|X|)] < \infty \tag{6}$$

hold.

P r o o f. The sufficiency of the conditions follows as in [3, Theorem T22, p. 19].

To prove the necessity of the condition, let $(\Omega', \mathcal{F}', P')$ be any non-atomic probability space. Then the uniform integrability of $\mathcal{H}$ implies the existence of a nonnegative random variable $Y \in L^1(\Omega', \mathcal{F}', P')$ such that $|X| \leqslant Y$ for all $X \in \mathcal{H}$, by [1, Theorem 4.2, p. 401]. Assume that Y is not essentially bounded. Since Y is integrable, the lemma entails the existence of a nonnegative increasing convex function $\Phi : R^+ \to R$ satisfying (3) and such that $E[\Phi(Y)] < \infty$. But $|X| \leqslant Y$ for all $X \in \mathcal{H}$ and so, by [2, Theorem 2.3, p. 1330], $E[\Phi(|X|)] \leqslant E[\Phi(Y)] < \infty$ for all $X \in \mathcal{H}$. Thus (6) follows.

Finally, if Y is essentially bounded, the result is trivial.

R e m a r k. In the above theorem, if $\mathcal{H}$ is uniformly integrable, the lemma shows that there exist (uncountably) infinitely many functions Φ which are nonnegative, increasing convex and satisfying both (3) and (6).

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