PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 25 (39), 1979, pp. 122—130

A SEQUENTIAL THEORY OF SOME SPACES OF GENERALIZED FUNCTIONS

E. Pap, S. Pilipović

(Communicated November 14, 1977)

1. Introduction

P. Antosik, J. Mikusiński and R. Sikorski have constructed in monography [1] a sequential theory of distributions. The intention of this theory is to make the distribution theory simple. Namely, in order to introduce distributions one can proceed in various ways. One of the earliest methods is the L. Schwartz's method of functionals. Actually there exists a large literature based on their description. A disadvantage of the functional approach is that it requires a deep knowledge of functional analysis. On the contrary, the sequential theory is based on the notions of sequences of functions and numbers. This helps to make the proofs of all theorems elementary and very simple.

The sequential theory of tempered and periodic distributions developed in [1] is of special interest.

In this paper, using the ideas of Zemanian from [3], we develop a sequential theory for some spaces $\mathcal{U}'$ of generalized functions. We obtain the orthonormal expansions of generalized functions from $\mathcal{U}'$ in the multidimensional case. As special cases of these spaces we obtain tempered distributions. Zemanian [3] has obtained orthonormal expansions in one dimensional case with Schwartz's approach to the theory. In our theory we define the generalized functions from $\mathcal{U}'$ do not using the notions of duals and families of semi-norms. So we do not need the space of special test functions for the definition of generalized functions and their convergence, from the space $\mathcal{U}'$. The multidimensional case has some differences to one dimensional case which requires some modifications.

The first part of the paper is concerned with the general theory of the spaces $\mathcal{U}'$. By an elementary theory of the Köthe spaces developed in [1] we prove in the second part of the paper, with some additional assumtions, the important theorem of the equivalence of weak and strong convergence.

2. Some notions and notations

In this paper we use terminology and notions from [1] and [3]. Now we shall give only those which are specific for this paper.

P^q is the set of all non-negative integer points of R^q (q-dimensional Euclidean space), and B^q is the set of all integer points of R^q.

Let $x=(\xi_1, \dots, \xi_q)$ be an element of R^q and $k=(k_1, \dots, k_q)$ be an element of B^q. Then we have

$$x^k=\xi_1^{k_1} \dots \xi_q^{k_q} \text{ and}$$

$$x^r=\xi_1^r \dots \xi_q^r, \text{ if } r \text{ is an integer.}$$

Let $L_2(I_1 \times \dots \times I_q)$ be the space of all complex valued locally integrable functions defined on the interval $I_1 \times \dots \times I_q \subset R^q$, where $I_i=(a_i, b_i)$, $i=1, \dots, q$, (I_i may also be the whole R) such that

$$\|f\|=\Big[\int_{I_1 \times \dots \times I_q} |f(x)|^2\, dx\Big]^{1/2} < \infty$$

holds.

Further, let $\{\psi_s^i\}$ be a complete orthonormal set in the space $L_2(I_i)$. We suppose that there exists a linear operator $\mathbf{R}_i: A_i \to A_i$, where A_i is the linear span of $\{\psi_1^i, \psi_2^i, \dots\}$, and such that there exists a sequence of real numbers $\{\lambda_{s,i}\}$ such that $|\lambda_{s,i}| \to \infty$ as $s \to \infty$ and that this sequence is not decreasing such that

$$\mathbf{R}_i \psi_s^i=\lambda_{s,i} \psi_s^i, \quad s=0, 1, \dots \; (i=1, \dots q).$$

In the special case we take, as in [3],

$$\mathbf{R}_i=\theta_{0,i} D_i^{\nu_i^1} \theta_{1,i} \dots D_i^{\nu_i^m} \theta_{m,i}, \quad i=1, \dots, q$$

where $D_i=d/d\xi_i$, the ν_i^k are positive integers, and the $\theta_{k,i}$ are smooth complex functions on I_i, which are different from zero on I_i and the $\theta_{k'i}$ and ν_i^k are such that

$$\mathbf{R}_i=\overline{\theta}_{m,i}(-D_i)^{\nu_i^m} \dots \overline{\theta}_{1,i}(-D_i)^{\nu_i^1} \overline{\theta}_{0,i}, \quad (\overline{\theta}_{k,i}(x)=\overline{\theta_{k,i}(x)})$$

holds and $\{\psi_s^i\}$ is a complete orthonormal set of smooth functions in $L_2(I_i)$.

If $x=(\xi_1, \dots, \xi_q) \in I_1 \times \dots \times I_q$ and $n=(\nu_1, \dots, \nu_q) \in P^q$ we define

$$\psi_n(x)=\psi_{\nu_1}^1(\xi_1) \dots \psi_{\nu_q}^q(\xi_q).$$

By E. Pap ([2]) the $\{\psi_n\}$ is an orthonormal complete set of functions in the space $L_2(I_1 \times \dots \times I_q)$.

On the linear span of $\{\psi_n\}$ we define $\mathbf{R}_i$

$$\mathbf{R}_i(\psi_n)=\psi^1_{\nu_1}\dots\psi^{i-1}_{\nu_{i-1}}\,\mathbf{R}(\psi^i_{\nu_i})\,\psi^{i+1}_{\nu_{i+1}}\dots\psi^2_{\nu_2} \text{ and } \mathbf{R}_i(\psi_n+\psi_m)=$$
$$=\mathbf{R}_i(\psi_n)+R_i(\psi_m)$$

3. Some spaces of generalized functions

Now, for the q-dimensional case we define

$$\mathbf{R}=\mathbf{R}_i\dots\mathbf{R}_q$$

It is easy to see that for the special case

$$\mathbf{R}=\theta_0\,D^{n_1}\theta_1\,D^{n_2}\dots D^{n_m}\theta_m$$

where

$\theta_k(x)=\theta_{k,1}(\xi_1)\dots\theta_{k,q}(\xi_q)$ for $x=(\xi_1,\dots,\xi_q)\in I_1\times\dots\times I_q$ and $n_k=(\nu^k_1,\dots\nu^k_q)\in P^q$ for $k=1,2,\dots,m$ and $D^{n_k}=D_1^{\nu^k_1}\dots D_q^{\nu^1_q}$.

So we have

$\mathbf{R}\,\psi_n=\lambda^1_n\psi_n$, $n=(\nu_1,\dots,\nu_q)\in P^q$, where $\lambda^1_n=\lambda_{\nu_1,1}\dots\lambda_{\nu_q,q}$. We alternatively write λ_n instead of λ^1_n if there is no possibility of misinterpretation.

Throughout the paper let A_ν be any sequence of finite subsets of P^q such that $A_\nu\subset A_{\nu+1}$ and $\lim\limits_{\nu\to\infty}A_\nu=P^q$.

We shall define a space of generalized functions depending on an orthonormal base $\{\psi_n\}$, $n\in P^q$, an operator $\mathbf{R}$ and a set $I_1\times\dots\times I_q\subset R^q$.

Definition 1. *A sequence* $\{\sum\limits_{n\in A_\nu}a_n\psi_n\}$ *is said to be* **R**-*fundamental if there exists a convergente sequence* $\{\sum\limits_{n\in A_\nu}c_n\psi_n\}$ *in* $L_2(I_1 x\dots x I_q)$ *and* $k\in P^q$ *such that* $\mathbf{R}^k\sum\limits_{n\in A_\nu}c_n\psi_n=\sum\limits_{\substack{n\in A_\nu\\ \lambda_n\neq0}}a_n\psi_n$ *for all* $\nu\in N$ *and*

$$\sum_{\lambda_n=0}|a_n|^2\,\tilde{\lambda}_n^{-2k}<\infty.\qquad \tilde{\lambda}_{\nu_i,i}=\begin{cases}1 & \text{if } \lambda_{\nu_i,i}=0\\ |\lambda_{\nu_i,i}| & \lambda_{\nu_i,i}\pm0.\end{cases}$$

We say that two **R**-fundamental sequences $\{\sum\limits_{n\in A_\nu}a_n\psi_n\}$ and $\{\sum\limits_{n\in B_\nu}b_n\psi_n\}$ are equivalent if $a_n=b_n$ for all $n\in P^q$.

The obtained equivalence classes will be called generalized functions from $\mathcal{U}'$. Element f from $\mathcal{U}'$, represented by the **R**-fundamental sequence $\{\sum\limits_{n\in A_\nu}a_n\psi_n\}$ from the definition 1., will be also denoted as

$$\mathbf{R}^kF+\sum_{\lambda_n=0}a_n\psi_n,\quad\text{where}\quad F\overset{2}{=}\sum_{n\in P^q}c_n\psi_n.$$

We identify any function from $L_2(I_1 \times \cdots \times I_q)$ with a generalized function represented by a **R**-fundamental sequence $\{\sum_{n\in A_\nu} a_n \psi_n\}$ such that $k=(0, \ldots 0)$ and $a_n=0$ for $\lambda_n=0$.

If $f\in\mathcal{U}'$ is represented by the **R**-fundamental sequence $\{\sum_{n\in A_\nu} a_n \psi_n\}$, then we define $\mathbf{R}f$ as an element from $\mathcal{U}'$ represented by the **R**-fundamental sequence $\{\mathbf{R}\sum_{n\in A_\nu} a_n \psi_n\}$.

Definition 2. *We say that a sequence of generalized functions f_n from $\mathcal{U}'$ strongly converges to $f\in\mathcal{U}'$ and we write $f_n \overset{\mathcal{U}'}{\to} f$, iff there exist square integrable functions F_n, F such that*

$$\mathbf{R}^k F_n + \sum_{\lambda_p=0} c_{np}\psi_p = f_n, \qquad \mathbf{R}^k F + \sum_{\lambda_p=0} c_p \psi_p = f$$

for some fixed $k\in P^q$ and

$$F_n \overset{2}{\to} F, \quad \sum_{\lambda_p=0} |c_{np}-c_p|^2 \tilde{\lambda}_p^{-2k} \to 0 \text{ as } n\to\infty.$$

It is obvious that $f_n \overset{\mathcal{U}'}{\to} f$ implies $\mathbf{R}^p f_n \overset{\mathcal{U}'}{\to} \mathbf{R}^p f$ for $p\in P^q$.

R-fundamental sequence $\{\sum_{n\in A_\nu} a_n \psi_n\}$ which represents the generalized function f, converges strongly to f and we write $f \overset{\mathcal{U}'}{=} \sum_{n\in P^q} a_n \psi_n$.

We alternatively write $=$ instead of $\overset{\mathcal{U}'}{=}$ if there is no possibility of misinterpretation.

Now, we can prove an important theorem which generalize the theorem 8.1.1. from [1].

Theorem 1. *If for some $k\in P^q$,*

$$(1) \qquad \sum_{n\in P^q} \tilde{\lambda}_n^{-2k} |a_n|^2 < \infty.$$

then there is a generalized function f from $\mathcal{U}'$ such that

$$(2) \qquad f = \sum_{n\in P^q} a_n \psi_n.$$

Conversely, if f is a generalized function from $\mathcal{U}'$, then there are numbers a_n satisfying (1) *such that* (2) *holds.*

Proof. Obviously

$$\sum_{\lambda_n \neq 0} \tilde{\lambda}_n^{-2k} |a_n|^2 < \infty$$

holds for some $k \in P^q$. But this means that there is a function $F \in L_2(I_1 \times \cdots \times I_q)$ with the expansion

$$F = \sum_{\lambda_n \in P^q} \lambda_n^{-k} a_n \psi_n \, (\lambda_n \neq 0).$$

So we have that

$$\mathbf{R}^k F + \sum_{\lambda_p = 0} a_p \psi_p = \sum_{n \in p^q} a_n \psi_n \tag{3}$$

and by the definition 1., the left part of (3) is a generalized function f from $\mathcal{U}'$.

Conversely if f is a generalized function from $\mathcal{U}'$ then there is a function $F \in L_2(I_1 \times \cdots \times I_q)$ such that (3) holds. But then we have $\lambda_n^k c_n = a_n$, $n \in P^q$, for $\lambda_n \neq 0$, $F = \sum_{\lambda_n \neq 0} c_n \psi_n$,

$$\sum_{\lambda_n \in P^q} |c_n|^2 < \infty$$

and (1) follows.

Definition 3. *We say that a complex valued function* $\varphi = \sum_{n \in P^q} a_n \psi_n$ *from* $L_2(I_1 \times \cdots \times I_q)$ *is an element of* $\mathcal{U}$ *iff for every* $k \in P^q$

$$\sum_{n \in P^q} \tilde{\lambda}_n^{2k} |a_n|^2 < \infty.$$

Definition 4. *Inner product of a generalized function* $f = \sum_{n \in P^q} a_n \psi_n$ *from* $\mathcal{U}'$ *and a function* $\varphi = \sum_{n \in P^q} b_n \psi_n$ *from* $\mathcal{U}$ *is*

$$(f, \varphi) = \sum_{n \in P^q} a_n \bar{b}_n$$

The preceding definition is consistent because the sum on the right side exists. These follows from the inequality

$$\sum_{n \in P^q} |a_n \bar{b}_n| \leqslant \Big(\sum_{n \in P^q} |\tilde{\lambda}_n^{-k} a_n|^2 \sum_{\lambda \in P^q} |\tilde{\lambda}_n^k \bar{b}_n|^2 \Big)^{1/2}.$$

We see that two generalized functions $f, g \in \mathcal{U}'$ are equal iff

$$(f, \varphi) = (g, \varphi)$$

for every $\varphi \in \mathcal{U}$.

Now we can prove easily

Theorem 2. *If* $f = \sum_{n \in P^q} a_n \psi_n \in \mathcal{U}'$ *then* $a_n = (f, \psi_n)$ *which implies that the expansion of the given generalized function is unique.*

4. Convergence in some spaces of generalized functions

Let a sequence $\{\tilde{\lambda}_n\}$ satisfy the following condition

$$\sum_{\nu_i=0}^{\infty} 1/\tilde{\lambda}_{i,\nu_i}^{r_i} < \infty \qquad (i=1, \ldots, q) \tag{4}$$

for some

$$r=(r_1, \ldots, r_q) \in P^q.$$

From now on we observe the spaces $\mathcal{U}'$ and $\mathcal{U}$ for which the condition (4) is satisfied, and in this case those spaces we denote with $\mathcal{U}_0'$ and $\mathcal{U}_0$.

Using the condition (4) we can prove easily

Theorem 3 *If for some* $k \in P^q$ *and* $M>0$ *for every* $n \in P^q$

$$|a_n| < M \tilde{\lambda}_n^k \tag{5}$$

is satisfied, there is a generalized function $f \in \mathcal{U}_0'$ *such that* f *is of the form* (2). *Conversely, if* f *is a generalized function from* $\mathcal{U}_0'$ *then there are numbers* a_n *satisfying* (5) *such that* f *is of the form* (2).

Proof. It is obvious that the condition (5) is necessary.

From (4) we know that

$$\sum_{n \in P^q} \frac{1}{\tilde{\lambda}_n^{2r}} = \prod_{i=1}^{q} \sum_{\nu_i=0}^{\infty} \frac{1}{\tilde{\lambda}_{\nu_i,i}^{2r_i}} < \infty \quad \text{for } r=(r_1, \ldots, r_q) \in P^q.$$

Using (5) we have

$$|a_n| \tilde{\lambda}_n^{-k-r} < M \tilde{\lambda}_n^{-r}.$$

It implies

$$\sum_{n \in P^q} \tilde{\lambda}_n^{-2k-2r} |a_n|^2 < \prod_{i=1}^{q} \sum_{\nu_i=0}^{\infty} \frac{M^2}{\tilde{\lambda}_{\nu_i,i}^{2r_i}}$$

and using theorem 1. we prove our assertion.

Theorem 4 *A sequence of generalized functions* $f_n = \sum_{p \in P^q} a_{np} \psi_p$ *from* $\mathcal{U}'$ *converges to* $f = \sum_{p \in P^q} a_p \psi_p$ *iff*

$$\sum_{p \in P^q} \tilde{\lambda}_p^{-2k} |a_{np} - a_p|^2 \to 0 \text{ as } n \to \infty. \tag{6}$$

Proof. Let

$$f_n = \mathbf{R}^k F_n + \sum_{\lambda_p=0} a_{np} \psi_p, \qquad F_n = \sum_{\lambda_R \neq 0} c_{np} \psi_p,$$

$$f = \mathbf{R}^k F + \sum_{\lambda_p=0} a_p \psi_p, \qquad F = \sum_{\lambda_p \neq 0} c_p \psi_p.$$

By Parseval formula we have

$$\int |F_n - F|^2 = \sum_{p\in P^q} |c_{np} - c_p|^2 \to 0$$

and from the equality $|a_{np} - a_p| = |\lambda_p^k (c_{np} - c_p)|$ for $\lambda_p \neq 0$, follows that the condition (6) is necessary and sufficient.

T h e o r e m 5. *A sequence of generalized functions f_n from $\mathcal{U}_0'$ converges to $f\in\mathcal{U}_0'$ iff*

$$a_{np} \to a_p \text{ as } n\to\infty \qquad (p\in P^q)$$

and moreover, there exist an index $k\in P^q$ and a number $M>0$ such that

(7) $$\tilde{\lambda}_n^{-k}\, |a_{np}| < M \text{ for all } n\in N \text{ and } p\in P^q.$$

P r o o f. From the preceding theorem follows that the conditions of this theorem are necessary.

Assume that $a_{np}\to a_p$ as $n\to\infty$ and that (7) holds for some k and M. From (7) follows that

$$\tilde{\lambda}_p^{-k} |a_p| < M \text{ for all } p\in P^q$$

and $\tilde{\lambda}_p^{-2k} |a_{np} - a_p|^2 < 4M^2$.

For any $\varepsilon>0$ there is a $p_0\in P^q$ such that

$$\sum_{p\in P^q} \tilde{\lambda}_p^{-2k-2r} |a_{np} - a_p|^2 \leqslant \sum_{p_0>p\in P^q} \tilde{\lambda}_p^{-2k-2r} |a_{np} - a_p|^2 + 4M^2 \sum_{p_0\not> p\in P^q} \tilde{\lambda}_p^{-2r}.$$

As the first sum on the right side is finite, it is less then $\varepsilon/2$ for sufficiently large n. We select p_0 such that $\sum_{p_0\not> p\in P^q} \tilde{\lambda}_p^{-2r} < \varepsilon/8M^2$ and the assertion follows.

Let $T_k = \{\tilde{\lambda}_p^k\}$ for $p\in P^q$ and $k=1, 2, \dots$.

The sequence $T_1, T_2, \dots$ has the following property

$$|T_k T_{k+1}| < \infty,$$

where $|A| = \sup_{p\in P^q} |a_p|$ for $A=\{a_p\}$, $p\in P^q$. So we can use the theory of Köthe spaces from [1].

$\overline{\mathcal{U}}_0$ is the space of all matrices $A=\{a_p\}$ such that

$$\|AT_k\| < \infty$$

for every $k\in N$ holds $(\|A\| = \sum_{p\in P^q} |a_p|)$.

We use also the note from [1, p. 227].

$\overline{\mathcal{U}}_0'$ is the space of all matrices $A=\{a_p\}$ such that

$$|AT_k^{-1}| < \infty$$

holds for some $k\in N$.

From the definition of $\overline{\mathcal{U}}_0$ follows that $\overline{\mathcal{U}}_0$ is identical with the coefficient matrices of elements from $\mathcal{U}_0$. In the same way, from theorem 3. follows that $\overline{\mathcal{U}}_0'$ is identical with the space of coefficient matrices of elements from $\mathcal{U}_0'$.

Using the theorem 5 we obtain

Theorem 6. *A sequence of generalized functions f_n from $\mathcal{U}_0'$ converges strongly to f iff the sequence of coefficient matrices of f_n converges strongly to the coefficient matrix of f.*

In the usual way we define that a sequence f_n from $\mathcal{U}_0'$ converges weakly to $f \in \mathcal{U}_0'$ iff for every $\varphi \in \mathcal{U}_0$

$$(f_n, \varphi) \to (f, \varphi) \text{ as } n \to \infty.$$

Let A_n, A and Φ be the coefficient matrices of $f_n, f \in \mathcal{U}_0'$ and $\varphi \in \mathcal{U}_0$. From the definition of (f, φ) follows that

$$(f_n, \varphi) = (A_n, \Phi) \text{ and } (f, \varphi) = (A, \Phi).$$

Theorem 7. *A sequence of generalized functions f_n from $\mathcal{U}_0'$ converges weakly to f iff the sequence of coefficient matrices of f_n converges weakly to the coefficient matrix of f.*

By the [1, Theorem 10. 8. 2.] we obtain

Theorem 8. *A sequence of generalized functions from $\mathcal{U}_0'$ converges strongly to f iff it converges weakly to f.*

Examples.

1. Examined cases ([1], [3]).

b) Tempered distributions. We take the Hermite functions

$$\psi_n(x) = h_n(x) = (2\pi)^{-\frac{q}{4}} \frac{1}{\sqrt{n!}} e^{-\frac{x^2}{4}} H(\xi_1) \dots H(\xi_q)$$

for $x = (\xi_1, \dots, \xi_q) \in R^q$ and $n = (\nu_1, \dots, \nu_q) \in P^q$, where $n! = \nu_1! \dots \nu_p!$, $x^2 = \xi_1^2 + \dots + \xi_q^2$ and

$$H_{\nu_i}(\xi_i) = (-1)^{\nu_i} e^{\frac{\xi_i^2}{2}} \left(e^{-\frac{\xi_i^2}{2}}\right)^{(\nu_i)} \text{ for } i = 1, \dots, q.$$

The differential operator is

$$\mathbf{R} = e^{\frac{x^2}{4}} D e^{-\frac{x^2}{2}} D e^{\frac{x^2}{4}} \text{ and}$$

$$\lambda_n = -n^1 \text{ for } n \in P^q.$$

The corresponding space $\mathcal{U}_0'$ in this case coincide with the space $\mathcal{S}'$ of tempered distributions. The construction in this paper differs from the construction of [1]. In the notations of [1] we have $\mathbf{R}=dD$ (D is the tempered derivative). Our approach is near to Zemanian's [3] (in one dimensional case), but he uses the Schwartz's approach to the theory of distributions.

b) Periodic generalized functions. We take Fourier functions

$$\psi_n(x)=(2\pi)^{-\frac{q}{2}}\, e^{inx}$$

for $x=(\xi_1, \ldots, \xi_q)\in(-\pi, \pi)^q$ and $n=(\nu_1, \ldots, \nu_q)\in B^q$, where $nx=\sum_{i=1}^{q}\nu_i\,\xi_i$ and B^q is the set of all integer points of R^q, the differential operator

$$\mathbf{R}=-i\,D \text{ and}$$

$$\lambda_n=n^1 \text{ for } n\in B^q.$$

2. We can take also the other orthonormal functions as: Laguerre's, Jacobi's. (Legendre's, Tchebishev's, Gegenbauer's), Bessel's.

For the construction of the spaces $\mathcal{U}_0'$ in this case we can use the constructions for one dimensional case of [3].

3. The functions $\psi^i_{\nu_i}$, $i=1, \ldots, q$ need not be of the same type. So we can take for example

$$\psi_n(x)=h_{\nu_1}(\xi_1)\,(2\pi)^{-1/2}\, e^{i\nu_2\xi_2}$$

for $x=(\xi_1, \xi_2)\in(-\infty, +\infty)\times(-\pi, \pi)$ and $n=(\nu_1, \nu_2)\in P\times B$, the differential operator

$$\mathbf{R}=-i\,e^{-\frac{\xi_1^2}{4}}\,D_1\,D_2\,e^{\frac{\xi_1^2}{2}}\,D_1\,e^{-\frac{\xi_1^2}{4}} \text{ and}$$

$$\lambda_n=-n^1, \quad n\in P\times B.$$

We are grateful to prof. D. Mitrović for his comment on our paper.

REFERENCES

[1] P. Antosik, J. Mikusiński, R. Sikorski, *Theory of distributions, The sequential approach*, Warszawa, 1973.

[2] E. Pap, *A note on orthogonal expansions in multidimensional case*, Pub. Inst. Math., t. 22 (36) (1977), 211—214.

[3] A. H Zemanian, *Integral transformations*, Interscience Publ., New York, 1968.