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LINEAR TRANSFORMATIONS ON MATRICES: THE INVARIANCE OF GENERALISED PERMUTATION MATRICES, II

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1. Introduction

Let F be a field, $M_n(F)$ be the vector space of all n-square matrices with entries in F and $\mathcal{U}$ a subset of $M_n(F)$. It is of interest to determine the structure of linear map $T: M_n(F) \to M_n(F)$ such that $T(\mathcal{U}) \subseteq \mathcal{U}$. For example, $\mathcal{U}$ is the set $\{X \in M_n(F): \det(X) = 0\}$ [1], the unitary group [2] or $GL_n(\mathbf{C})$, the group of all nonsingular n-square matrices over $\mathbf{C}$ [4]. Other results in this direction can be found in [3]. In this note we consider $\mathcal{U}$ to be a set of generalized permutation matrices relative to some permutation set and with entries in some nontrivial subgroup of F^* where F^* is the multiplicative group of F. More precisely we let S_n be the symmetric group acting on the set $\{1, 2, \ldots, n\}$ and if S is a subset of F we define

$$\Gamma_n(S) = \left\{ \alpha = \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in S \right\}.$$

where e_i denote the n-column vector with 1 at the i-th row and zero elsewhere. If $\alpha \in \Gamma_n(F^*)$ and $\sigma \in S_n$ then $P(\sigma, \alpha)$ is the matrix whose (i, j) entry is $\alpha_i \delta_{i\sigma^{-1}(j)}$ (where $\delta_{ij} = 1$ if $i = j$ and 0 elsewhere) and we call $P(\sigma, \alpha)$ a generalized permutation matrix. If G is nonempty subset of S_n and H is a subgroup of F^* we define

$$P(G, H) = \{P(\sigma, \alpha). \quad \alpha \in \Gamma_n(H), \quad \sigma \in G\},$$

$$\mathcal{T}P(G, H) = \{T : T \text{ is a linear transformation of } M_n(F) \text{ to itself and } T(P(G, H)) = P(G, H)\},$$

$$\& P(G, H) = \{T : T \text{ is a linear transformation of } M_n(F) \text{ to itself and } T(P(G, H)) \subseteq P(G, H)\}.$$

It is trivial that $\& P(G, H)$ is a multiplicative semigroup and $\mathcal{J}P(G, H) \subseteq \subseteq \& P(G, H)$. Let

$$\mathcal{H} = \{H : H \text{ is a subgroup of } F^* \text{ and there do not exist}$$
$$a, b \in F^* \text{ such that } Ha + b \subseteq H\}.$$

The set $\mathcal{H}$ is nonempty. For examples, F^* is in $\mathcal{H}$ for every field F, every nontrivial finite subgroup of F^* is in $\mathcal{H}$ and every subgroup H of the unit circle of the complex plane and $|H| > 2$ is in $\mathcal{H}$ [5]. But if $\mathbf{R}$ is the real field, then $\mathbf{R}^+ = \{x \in \mathbf{R} : x > 0\}$ is not in $\mathcal{H}$ since $\mathbf{R}^+ + 1 \subseteq \mathbf{R}^+$. Our main result is a characterization of all linear maps in $\& P(G, H)$ where G is a regular subset of S_n, i.e., for every pair (i, j), $1 \leqslant i, j \leqslant n$ there exists exactly $\sigma \in G$ such that $\sigma(i) = j$, and H is a nontrivial group in $\mathcal{H}$. If G is a regular subset, a doubly transitive subset (subgroup) of S_n and H is a nontrivial group in $\mathcal{H}$, then $\mathcal{J}P(G, H)$ has been characterized and the structure of the group $\mathcal{J}P(G, H)$ has been obtained [5].

2. Main result

For $\alpha = \sum_{i=1}^{n} \alpha_i e_i \in \Gamma_n(F)$ and $1 \leqslant p \leqslant n$ we define $C(p, \alpha)$ to be the $n \times n$ matrix whose only nonzero entries are in p-th column, i.e., α_i at (i, p) position, $i = 1, 2, \dots, n$ and zero elsewhere. If S is a subset of F we define

$$C(p, S) = \{C(p, \alpha): \quad \alpha \in \Gamma_n(S)\},$$

$$C(\cdot, S) = \{C(p, \alpha): \quad 1 \leqslant p \leqslant n, \quad \alpha \in \Gamma_n(S)\}.$$

Let E_{ij} be the $n \times n$ matrix with 1 at (i, j) position and zero elsewhere. Let $G = \{g_1, g_2, \dots, g_n\}$ be a regular subset of S_n, H be a nontrivial subgroup of F^* and $f: M_n(F) \to M_n(F)$ be a linear map defined by

$$f(E_{ig_j(i)}) = E_{ij}, \quad 1 \leqslant i, j \leqslant n.$$

f is well-defined since G is regular. The map f straightens out the diagonals corresponding to the g_i into columns and hence $T \in \& P(G, H)$ if and only if $f \circ T \circ f^{-1}(C(\cdot, H)) \subseteq C(\cdot, H)$. Let $T_1 = f \circ T \circ f^{-1}$ and for $i = 1, 2, \dots, n$ let $h_i: \Gamma_n(F) \to C(i, F)$ be defined by

$$h_i(e_j) = E_{ji}, \quad j = 1, 2, \dots, n.$$

Then we have

Lemma 1. *$T \in \& P(G, H)$ if and only if $T_1 \circ h_i(\Gamma_n(H)) \subseteq C(\cdot, H)$ for each $i = 1, 2, \dots, n$.*

In this note it is shown that $T \in \& (P(G, H)$ only if for each $i = 1, 2, \dots, n$ either all images of $T_1 \circ h_i$ have the same nonzero column say j-th column (in which case $h_j^{-1} \circ T_1 \circ h_i$ is a linear map from $\Gamma_n(F)$ to $\Gamma_n(F)$ leaving $\Gamma_n(H)$

invariant) or there are two nonzero n-columns. The later case can happen only if $|H|=2$. More precisely we have the

Theorem *Let G be a regular subset of S_n and H be a nontrivial group in $\mathcal{H}$. Then $T\in \&P(G, H)$ if and only if for each $j=1, 2, \ldots, n$ there exist integers $1\leqslant r_j\leqslant n$, $1\leqslant p_j\leqslant n$, $1\leqslant i_{j1}<i_{j2}<\cdots<i_{jr_j}\leqslant n$ and a partition $\{\Omega_{j1}, \Omega_{j2}, \ldots, \Omega_{jr_j}\}$ of $\{1, 2, \ldots, n\}$ such that*

$$T_1(E_{i_{jk}j})=\sum_{l\in\Omega_{jk}}\gamma_{kj}^l E_{lp_j},\quad k=1, 2, \ldots, r_j$$

where $\gamma_{kj}^l\in H$ and

$$T_1(E_{kj})=0\quad \text{if}\quad k\notin\{i_{j1}, i_{j2}, \ldots, i_{jr_j}\}$$

or $r_j=2$ (this case can happen only if $|H|=2$) and there exist $1\leqslant i_{j1}, i_{j2}, p_j, q_j\leqslant n$, $i_{j1}\neq i_{j2}, p_j\neq q_j$; $\alpha_j, \beta_j\in\Gamma_n(H)$ such that

$$T_1(E_{i_{j1}j})=\frac{1}{2}C(p_j, \alpha_j)+\frac{1}{2}C(q_j, \beta_j),$$

$$T_1(E_{i_{j2}j})=\frac{1}{2}C(p_j, \alpha_j)-\frac{1}{2}C(q_j, \beta_j)$$

and $T_1(E_{kj})=0$ if $k\neq i_{j1}, i_{j2}$.

3 Proof In the following let $T\in\&P(G, H)$ where G is a regular subset of S_n.

Lemma 2 *Suppose H is a subgroup of F^* and $|H|>2$. Then for $j=1, 2, \ldots, n$ there exists integer $1\leqslant p_j\leqslant n$ such that*

$$T_1\circ h_j(e_k)\in C(p_j, F),\quad k=1, 2, \ldots, n.$$

Proof. Let ξ, η be distinct elements in H and both are distinct from 1. By Lemma 1, for $1\leqslant k\leqslant n$, there exist integers $1\leqslant p_j, q_{jk}, r_{jk}\leqslant n$ and $\alpha_j, \beta_{jk}, \gamma_{jk}\in\Gamma_n(H)$ such that

$$T_1\circ h_j\left(\sum_{i=1}^n e_i\right)=C(p_j, \alpha_j),\tag{3.1}$$

$$T_1\circ h_j(\xi e_k+\sum_{i\neq k}e_i)=C(q_{jk}, \beta_{jk}),$$

$$T_1\circ h_j(\eta e_k+\sum_{i\neq k}e_i)=C(r_{jk}, \gamma_{jk}).$$

Hence

$$T_1\circ h_j((1-\xi)e_k)=C(p_j, \alpha_j)-C(q_{jk}, \beta_{jk}),$$

$$T_1\circ h_j((1-\eta)e_k)=C(p_j, \alpha_j)-C(r_{jk}, \gamma_{jk})\ \text{or}$$

$$(1-\xi)^{-1}(C(p_j\alpha_j)-C(q_{jk}, \beta_{jk}))=(1-\eta)^{-1}(C(p_j, \alpha_j)-C(r_{jk}, \gamma_{jk}))\ \text{or}$$

$$((1-\xi)^{-1}-(1-\eta)^{-1})C(p_j, \alpha_j)=(1-\xi)^{-1}C(q_{jk}, \beta_{jk})-(1-\eta)^{-1}C(r_{jk}, \gamma_{jk}),$$

If $q_{jk} \neq r_{jk}$ then the matrix of the right hand side has two nonzero columns and that of the left hand side only one nonzero column, which is impossible. Hence $p_j = q_{jk} = r_{jk}$ and

$$T_1 \, oh_j(e_k) = (1-\xi)^{-1}(C(p_j, \alpha_j) - C(p_j, \beta_{jk})) = C(p_j, \theta_{jk})$$

where $\theta_{jk} \in \Gamma_n(F)$, Note that p_j is uniquely determined by (3.1) for each j.

Lemma 3 *Suppose $|H| = 2$. Then for each $j = 1, 2, \dots, n$ either*

(i) *there exists integer $1 \leqslant p_j \leqslant n$ such that*

$$T_1 \, oh_j(e_i) = C(p_j, \theta_{ji}), \quad i = 1, 2, \dots, n$$

where $\theta_{ji} \in \Gamma_n(F)$ or

(ii) *there exist integers $1 \leqslant p_j$, q_j i_{j1}, $i_{j2} \leqslant n$, $i_{j1} \neq i_{j2}$ $p_j \neq q_j$ and α_j, $\beta_j \in \Gamma_n(H)$ such that*

$$T_1 \, oh_j(e_{i_{j1}}) = \frac{1}{2} C(p_j, \alpha_j) + \frac{1}{2} C(q_j, \beta_j),$$

$$T_1 \, oh_j(e_{i_{j2}}) = \frac{1}{2} C(p_j, \alpha_j) - \frac{1}{2} C(q_j, \beta_j),$$

and

$$T_1 \, oh_j(e_l) = 0 \quad \text{for} \quad l \neq i_{j1}, \, i_{j2}, \; 1 \leqslant l \leqslant n.$$

Proof. Let $\mu_0 = \sum_{k=1}^{n} e_k$ and $\mu_i = \sum_{k \neq i} e_k - e_i$, $i = 1, 2, \dots, n$.

Then $\mu_i \in \Gamma_n(H)$ and by Lemma 1, there exist integers $1 \leqslant p_j$, $q_{ji} \leqslant n$, $i = 1, 2, \dots, n$ and α_j, β_{j1}, $\beta_{j2}, \dots, \beta_{jn} \in \Gamma_n(H)$ such that

$$T_1 \, oh_j(\mu_0) = C(p_j, \alpha_j),$$

$$T_1 \, oh_j(\mu_i) = C(q_{ji}, \beta_{ji}), \; i = 1, 2, \dots, n.$$

Hence

$$2 \, T_1 \, oh_j(e_i) = C(p_j, \alpha_j) - C(q_{ji}, \beta_{ji}), \; i = 1, 2, \dots, n.$$

It follows from $|H| = 2$ that char $F \neq 2$ and

$$T_1 \, oh_j(e_i) = \frac{1}{2} C(p_j, \alpha_j) - \frac{1}{2} C(q_{ji}, \beta_{ji}), \; i = 1, 2, \dots, n.$$

If $q_{ji} = p_j$ for $i = 1, 2, \dots, n$ then $T_1 \, oh_j(e_i) = \frac{1}{2} C(p_j, \alpha_j - \beta_{ji})$, where $\alpha_j - \beta_{ji} \in \Gamma_n(F)$, $i = 1, 2, \dots, n$.

If $q_{ji} \neq p_j$ for some i, say $i=1$, then for $l=2, 3, \ldots, n$ since

$$A = \sum_{k \neq 1, l} T_1 \, oh_j (e_k) - T_1 \, oh_j (e_1) - T_1 \, oh_j (e_l)$$
$$= T_1 \, oh_j (\mu_l) - 2 \, T_1 \, oh_j (e_1)$$
$$= C(q_{j1}, \beta_{j1}) - C(p_j, \alpha_j) + C(q_{jl}, \beta_{jl})$$

is in $C(\cdot, H)$, it follows that either $C(p_j, \alpha_j) = C(q_{jl}, \beta_{jl})$, i.e.,

$$T_1 \, oh_j (e_l) = 0 \text{ or } C(q_{jl}, \beta_{jl}) = -C(q_{j1}, {}_{j1}), \text{ i.e.,}$$

$$T_1 \, oh_j (e_l) = \frac{1}{2} C(p_j, \alpha_j) + \frac{1}{2} C(q_{j1}, \beta_{j1}).$$

If r is the number of nonzero matrices among $T_1 \, oh_j (e_i)$, $i=1, 2, \ldots, n$, then without loss of generality we may assume that for $i=1, 2, \ldots, r$, $T_1 \, oh_j (e_i) \neq 0$. Now

$$T_1 \, oh_j (\mu_0) = \frac{r}{2} C(p_j, \alpha_j) + \frac{r-2}{2} C(q_{j1}, \beta_{j1})$$

which is in $C(\cdot, H)$ only if $r=2$, $r = \text{char } F$ or $r-2 = \text{char } F$. If $r = \text{char } F$ then $r \geqslant 3$ and it follows that

$$T_1 \, oh_j (\mu_2) = \frac{r-2}{2} C(p_j, \alpha_j) + \frac{r-4}{2} C(q_{j1}, \beta_{j1})$$

which is not in $C(\cdot, H)$, a contradiction. If $r-2 = \text{char } F$, then $r \geqslant 5$ and it follows that

$$T_1 \, oh_j \Big(\sum_{i \neq 2, 3} e_i - e_2 - e_3 \Big) = \frac{r-4}{2} C(p_j, \alpha_i) + \frac{r-6}{2} C(q_{j1} \, \beta_{j1})$$

which is not in $C(\cdot, H)$, again a contradiction. Hence $r=2$ and the Lemma follows.

Lemma 4 *Suppose* $H \in \mathscr{H}$ *and* H *is nontrivial,* $T_1 \, oh_j (e_i) = C(p, \theta_i)$, $\theta_i = (\theta_{i1}, \theta_{i2}, \ldots, \theta_{in}) \in F_n(F)$ *and* $\Omega_i = \{l : \theta_{il} \neq 0\}$, $i = 1, 2, \ldots, n$. *Then* $\Omega_i \cap \Omega_k = \varnothing$ *if* $i \neq k$, $1 \leqslant i, k \leqslant n$, $\bigcup_{i=1}^{n} \Omega_i = \{1, 2, \ldots, n\}$ *and* $\theta_i \in \Gamma_n (H \cup \{0\})$.

Proof. Suppose $l \in \Omega_i$ for $1 \leqslant i \leqslant r$ where $r \geqslant 2$. Then we may choose $\alpha_2, \alpha_3 \ldots, \alpha_n \in H$ so that $\sum_{i=2}^{n} \alpha_l \theta_{il} \neq 0$. Let

$$a = \theta_{1l}, \; b = \sum_{l=2}^{n} \alpha_l \theta_{il}.$$

Then for $\alpha_1 \in H$, $T_1 \, oh_j \left(\sum_{i=1}^{n} \alpha_i e_i \right)$ is in $C(\cdot, H)$, i.e., $\alpha, a+b \in H$. Hence $Ha + b \subseteq H$, a contradiction. Since T_1 preserves $C(\cdot, H)$ it follows that $\theta_i \in \Gamma_n (H \cup \{0\})$ and $\bigcup_{i=1}^{n} \Omega_i = \{1, 2, \ldots, n\}$.

Proof of Theorem. The theorem follows by Lemmas 1, 2, 3 and 4.

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