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THE ANALYSIS OF A CLASS OF EXPONENTIAL OPERATIONAL FUNCTIONS

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Abstract. The main purpose of this paper is to investigate the exponential function

$$\exp\left[-\lambda\left(s^{\alpha}+as^{-\beta}\right)^{\frac{1}{\gamma}}\right] \tag{1}$$

where s the differential operator in the field M of Mikusinski's operators, the coefficient $a>0$, α, β, γ are rational numbers such that $\alpha+\beta>0$ and

$$s^{-\alpha}=\left\{\frac{t^{\alpha-t}}{\Gamma(\alpha)}\right\} \text{ for } \alpha>0 \text{ and } s^{0}=1$$

The operational function (1) can be used in the application of Operational Calculus in Physics [6].

This paper consists of three parts. The first part describes the character of the exponential function (1). In the second, we give the representation of this function by means of convergent power series if $\frac{\alpha}{\gamma}<1$ and $\alpha+\beta>\frac{\alpha}{\gamma}$. In the third part we give the approximation of this function by the method which gave B. Stanković [9] [10].

The analysis of the exponential function (1) requires:

(i) To establish the existence of the exponential function (1).

(ii) If the exponential function (1) exists it remains to analyse its character: is it an element of class C, $\mathscr{L}$, a distribution or only an operational function*.

* C is the ring of continuous complex valued functions defined over $[0, \infty)$ with the operation sum and finit convolution. $\mathscr{L}$ is the ring of locally integrable functions over $[0, \infty)$ with the same operation. The quotient field of these rings is the field M of Mikusiński operators.

We denote by w the following operator

$$w=(s^{\alpha}+as^{-\beta})^{\frac{1}{\gamma}}=s^{\frac{\alpha}{\gamma}}(1+as^{-(\alpha+\beta)})^{\frac{1}{\gamma}}=s^{\frac{\alpha}{\gamma}}\sum_{k=0}^{\infty}\binom{\frac{1}{\gamma}}{k}a^k s^{-(\alpha+\beta)k}$$

$$=s^{\frac{\alpha}{\gamma}}+\frac{1}{\gamma}as^{\frac{\alpha}{\gamma}-(\alpha+\beta)}+\binom{\frac{1}{\gamma}}{2}a^2 s^{\frac{\alpha}{\gamma}-2(\alpha+\beta)}+\dots$$

Let $\frac{\alpha}{\gamma}-(n+1)(\alpha+\beta)$ be the greatest of the negative exponents. Then

$$f=\binom{\frac{1}{\gamma}}{n+1}a^{n+1}l^{(\alpha+\beta)(n+1)-\frac{\alpha}{\gamma}}+\binom{\frac{1}{\gamma}}{n+2}a^{n+2}l^{(\alpha+\beta)(n+2)-\frac{\alpha}{\gamma}}+\dots$$

where l is the integral operator, is a function continuous in $(0, \infty)$ and integrable in the neigbourhood of $t=0$.

So the operator w we can write in the form

$$w=s^{\frac{\alpha}{\gamma}}+\frac{1}{\gamma}as^{\frac{\alpha}{\gamma}-(\alpha+\beta)}+\dots+\binom{\frac{1}{\gamma}}{n}a^n s^{\frac{\alpha}{\gamma}-n(\alpha+\beta)}+f \tag{2}$$

If $\frac{\alpha}{\gamma}>0$ the operator (2) is reduced to the function $f\in C$. In the last case the exponential function (1) does exist.

Namely, in the case $\frac{\alpha}{\gamma}<0$ and $\alpha+\beta>0$ the operator w (2) is a function of class C also a logarithm and the exponential function (1) for every complex value of λ can be expanded in a convergent power series of the following form

$$\exp[-\lambda(s^{\alpha}+as^{-\beta})^{\frac{1}{\gamma}}]=1+\sum_{j=1}^{\infty}\frac{(-1)^j\lambda^j}{j!}s^{\frac{\alpha}{\gamma}j}(1+al^{\alpha+\beta})^{\frac{j}{\gamma}}$$

$$=1+\sum_{j=1}^{\infty}\frac{(-1)^j\lambda^j}{j!}\sum_{n=0}^{\infty}\binom{\frac{j}{\gamma}}{n}a^n l^{(\alpha+\beta)n-\frac{\alpha}{\gamma}j}$$

$$=1+\sum_{j=1}^{\infty}\frac{(-1)^j\lambda^j}{j!}\sum_{n=0}^{\infty}\frac{j(j-\gamma)}{\gamma^n}\,\frac{(j-2\gamma)\cdots(j-n\gamma+\gamma)}{n!}a^n l^{(\alpha+\beta)n-\frac{\alpha}{\gamma}j}$$

Substituting $n-j$ for n we obtain

$$=1+\sum_{j=1}^{\infty}\frac{(-1)^j\lambda^j}{(j-1)!}\sum_{n=j}^{\infty}\frac{(j-\gamma)\cdots(j-n\gamma+j\gamma+\gamma)}{\gamma^{n-j}(n-j)!}a^{n-j}\,l^{(\alpha+\beta)(n-j)-\frac{\alpha}{\gamma}j}$$

Changing the order of summation we obtain

$$\exp[-\lambda(s^{\alpha}+as^{-\beta})^{\frac{1}{\gamma}}]=1+\underset{\alpha\beta\gamma}{g}(\lambda)$$

where

$$\underset{\alpha\beta\gamma}{g}(\lambda)=\{\underset{\alpha\beta\gamma}{g}(\lambda, t)\}=$$

$$=\sum_{n=1}^{\infty}\left(\frac{a}{\gamma}\right)^{n}l^{(\alpha+\beta)n}\sum_{j=1}^{n}(-1)^j\left(\frac{\lambda\gamma}{a}\right)^j\frac{(j-\gamma)\cdots(j-n\gamma+j\gamma+\gamma)}{(n-j)!\,(j-1)!}s^{\left(\alpha+\beta+\frac{\alpha}{\gamma}\right)j}$$

is a parametric function and uniformly converges (in the usual sense) in every domain

$$|\lambda|\leqslant\lambda_0,\quad 0\leqslant t\leqslant T$$

Suppose that $\frac{\alpha}{\gamma}>0$.

In order that the exponential function

$$\exp\left\{\left[-s^{-\frac{\alpha}{\gamma}}-\frac{1}{\gamma}as^{\frac{\alpha}{\gamma}-(\alpha+\beta)}-\cdots-\binom{\frac{1}{\gamma}}{n}a^n s^{\frac{\alpha}{\gamma}-n(\alpha+\beta)}-f\right]\lambda\right\}$$

exists it is sufficient and necessary that $\exp(-\lambda s^{\frac{\alpha}{\gamma}})$ *exists* ([1] p. 442 and [2] p. 224)

But we know that $\exp(-\lambda s^{\frac{\alpha}{\gamma}})$ exists if $\frac{\alpha}{\gamma}<1$ and does not exist if $\frac{\alpha}{\gamma}>1$. In case $\alpha=\gamma$ and λ is real, the function $\exp(-\lambda s)$ represents the known translation operator and the series for the function $\exp(-\lambda s)$ is not convergent.

Also our exponential function $\exp(-\lambda w)$ where w is defined by the relation (2) contains in reality operational functions of two forms:

$$\exp(-\lambda s^{\frac{\alpha}{\gamma}}),\quad 0<\frac{\alpha}{\gamma}<1 \text{ and } \exp(-\lambda f)$$

where

$$f=\sum_{i=n+1}^{\infty}\binom{\frac{1}{\gamma}}{i}a^i l^{(\alpha+\beta)i-\frac{\alpha}{\gamma}}, \quad (\alpha+\beta)(n+1)-\frac{\alpha}{\gamma}>0$$

In the first part we will describe the character of the exponential function (1).

1. The character of the exponential function (1)

The character of the exponential function $\exp(-\lambda s^{\frac{\alpha}{\gamma}})$, $0<\frac{\alpha}{\gamma}<1$ is analysed in [3] [11] and [12].

1° if $0<\frac{\alpha}{\gamma}<1$ and $|\arg\lambda|<\frac{\pi}{2}\left(1-\frac{\alpha}{\gamma}\right)$ then

$$\exp(-\lambda s^{\frac{\alpha}{\gamma}})=\begin{cases} t^{-1}\Phi\left(0,\ -\frac{\alpha}{2};\ -\lambda t^{-\frac{\alpha}{\gamma}}\right), & t\neq 0\\ 0 & , \ t=0\end{cases}$$

where Φ is the function of E. M. Wright [8]. In this case $\exp(-\lambda s^{\frac{\alpha}{\gamma}})$ is from C. For she special cases $\frac{\alpha}{\gamma}=1/2$ and $\frac{\alpha}{\gamma}=2/3$ see [8] p. 115—116.

2° If $0<\frac{\alpha}{\gamma}<1$ and $|\arg\lambda|=\frac{\pi}{2}\left(1-\frac{\alpha}{\gamma}\right)$ then is:

$$\exp(-\lambda s^{\frac{\alpha}{\gamma}})=sl^{\frac{1}{2}}\left\{t^{-1/2}\Phi\left(1/2,\ -\frac{\alpha}{\gamma};\ -|\lambda|\, e^{\pm\frac{\pi}{2}\left(1-\frac{\alpha}{\gamma}\right)i}\, t^{-\frac{\alpha}{\gamma}}\right)\right\}$$

If $0<\frac{\alpha}{\gamma}\leqslant 2/3$ this exponential function is a distribution which is not a function and for $2/3<\frac{\alpha}{\gamma}<1$ it is a function which is not Lebesgue-integrable over $[0, T]$, $T>0$.

3° If $0<\frac{\alpha}{\gamma}<1$ and $\pi\geqslant|\arg\lambda|>\frac{\pi}{2}\left(1-\frac{\alpha}{\gamma}\right)$ then $\exp(-\lambda s^{\frac{\alpha}{\gamma}})$ is an operational function which is neither a distribution nor a function.

4° Let us consider the function $\exp(-\lambda f)$, $f\in\mathscr{L}$. Then $[\exp(-\lambda f)-1]\in\mathscr{L}$ i.e. it is a locally integrable function.

In special case $f = l^a$, $a > 0$, λ complex number, we have:

$$\exp(-\lambda l^a) = 1 + s\left(\sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k l^{ak+1}}{k!}\right)$$

$$= 1 + s\left\{\sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k t^{ak}}{k!\,\Gamma(ak+1)}\right\}$$

$$= 1 + \left\{\sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k t^{ak-1}}{\Gamma(k+1)\,\Gamma(ak)}\right\} = 1 + \{t^{-1}\,\Phi(0, a; -\lambda t^a)\}$$

where Φ is the function of E. M. Wright.

In the case $0 < a < 1$, $\exp(-\lambda l^a) - 1$ is from $\mathcal{L}$ and in the case $a \geqslant 1$ $\exp(-\lambda l^a) - 1$ is from C. If $a = 1$ then for every $k > -1$ we obtain

$$\exp(-\lambda l) = s^{k+1}\left\{\left(\frac{t}{\lambda}\right)^{k/2} J_k\left(2\sqrt{\lambda t}\right)\right\} = 1 - \left\{\sqrt{\frac{\lambda}{t}}\, J_1\left(2\sqrt{\lambda t}\right)\right\}$$

where $J_n(t)$ is the Bessel function of order n.

Also, the required exponential function (1) is of the following form

$$\exp[-\lambda(s^\alpha + as^{-\beta})^{\frac{1}{\gamma}}] = \exp(-\lambda s^{\frac{\alpha}{\gamma}}) \exp\left(-\frac{a}{\gamma} s^{\frac{\alpha}{\gamma}-(\alpha+\beta)} \lambda\right) \cdots$$

$$\cdot \exp\left[-\lambda \binom{\frac{1}{\gamma}}{n} a^n s^{\frac{\alpha}{\gamma}-(\alpha+\beta)n}\right] \exp(-\lambda f)$$

We can expand $\exp(-\lambda f)$ into the power series of λ by writing $\exp(-\lambda f) = 1 - \frac{\lambda}{1!} f + \frac{\lambda^2}{2!} f^2 - \ldots$. Neglecting the terms which do not appear explicitly on the right side of this formula, we obtain an approximation which is satisfactory for small values of λ and t.

2. The representation of the function (1) by means of convergent power series in the case $0 < \frac{\alpha}{\gamma} < 1$ and $\alpha + \beta > \frac{\alpha}{\gamma}$

In [4] [5] it has been given the representation of the special exponential functions. Now, if we denote by w_1 the following operator

$$w_1 = s^{\frac{\alpha}{\gamma}} - (s^\alpha + as^{-\beta})^{\frac{1}{\gamma}} = s^{\frac{\alpha}{\gamma}} [1 - (1 + al^{\alpha+\beta})^{\frac{1}{\gamma}}]$$

$$= -\sum_{n=0}^{\infty} \binom{\frac{1}{\gamma}}{n+1} a^{n+1}\, l^{(\alpha+\beta)(n+1)-\frac{\alpha}{\gamma}}$$

then w_1 is a function of class $\mathcal{L}$ also a logarithm.

Hence, the exponential function $\exp(\lambda w_1)$ can be expanded in a convergent power series:

$$\exp\{[s^{\frac{\alpha}{\gamma}} - (s^{\alpha} + as^{-\beta})^{\frac{1}{\gamma}}]\,\lambda\} = \sum_{i=0}^{\infty} \frac{(-1)^i \lambda^i}{i!} s^{\frac{\alpha}{\gamma} i} [(1 + al^{\alpha+\beta})^{\frac{1}{\gamma}}) - 1]^i =$$

$$= 1 + \sum_{i=0}^{\infty} \frac{(-1)^{i+1} \lambda^{i+1}}{(i+1)!} s^{\frac{\alpha}{\gamma}(i+1)} [(1 + al^{\alpha+\beta})^{\frac{1}{\gamma}} - 1]^{i+1}$$

But from the formula

$$(1 + al^{\alpha+\beta})^{\frac{1}{\gamma}} - 1 = \sum_{n=0}^{\infty} \binom{\frac{1}{\gamma}}{n+1} a^{n+1}\, l^{(\alpha+\beta)(n+1) - \frac{\alpha}{\gamma}}$$

we easily find, by induction, the expansion

$$[(s^{\alpha} + as^{-\beta})^{\frac{1}{\gamma}} - s^{\frac{\alpha}{\gamma}}]^{i+1} = s^{\frac{\alpha}{\gamma}(i+1)} [(1 + al^{\alpha+\beta})^{\frac{1}{\gamma}} - 1]^{i+1} =$$

$$= (al^{\alpha+\beta-\frac{\alpha}{\gamma}})^{n+1} \sum_{n=0}^{\infty} \binom{\frac{i+1}{\gamma}}{n+i+1} a^n\, l^{(\alpha+\beta)n} \qquad (i = 0, 1, 2, \ldots)$$

In view of this we can finally write

$$\text{(3)} \qquad \exp\{[s^{\frac{\alpha}{\gamma}} - (s^{\alpha} + as^{-\beta})^{\frac{1}{\gamma}}]\,\lambda\} = 1 +$$

$$+ \sum_{i=0}^{\infty} \frac{(-1)^{i+1} \lambda^{i+1}}{(i+1)!} l^{\left(\alpha+\beta-\frac{\alpha}{\gamma}\right)(i+1)} \sum_{n=0}^{\infty} \binom{\frac{i+1}{\gamma}}{n+i+1} a^{n+1+i}\, l^{(\alpha+\beta)i}$$

$$= 1 -$$

$$- \lambda a l^{\alpha+\beta-\frac{\alpha}{\gamma}} \sum_{i=0}^{\infty} \frac{(-1)^i \lambda^i}{i!} s^{\frac{\alpha}{\gamma} i} \sum_{n=0}^{\infty} \frac{(i+1-\gamma)\cdots(i+1-n\gamma-i\gamma)}{\gamma^{n+i+1}(n+i+1)!} a^{n+i}\, l^{(\alpha+\beta)(n+i)}$$

$$= 1 - \lambda a l^{\alpha+\beta-\frac{\alpha}{\gamma}}$$

$$\sum_{i=0}^{\infty} \frac{(-1)^i \lambda^i s^{\frac{\alpha}{\gamma} i}}{i!} \sum_{\nu=i}^{\infty} \frac{(i+1-\gamma)(i+1-2\gamma)\cdots(i+1-\nu\gamma)}{\gamma^{\nu+1}(\nu+1)!} a^{\nu}\, l^{\nu(\alpha+\beta)}$$

$$= 1 - \frac{\lambda a}{\gamma} l^{\alpha+\beta-\frac{\alpha}{\gamma}}$$

$$\sum_{\nu=0}^{\infty} \left(\frac{al^{\alpha+\beta}}{\gamma}\right)^{\nu} \frac{1}{(\nu+1)!} \sum_{i=0}^{\nu} \frac{(-1)^i \lambda^i (i+1-\gamma)(i+1-2\gamma)\cdots(i+1-\nu\gamma)}{i!} s^{\frac{\alpha}{\gamma} i}$$

$$= 1 + f_{\alpha\beta\gamma}(\lambda)$$

where

$$f_{\alpha\beta\gamma}(\lambda)=\{f_{\alpha\beta\gamma}(\lambda,t)\}=\frac{-\lambda a l}{\gamma}^{\alpha+\beta-\frac{\alpha}{\gamma}} \tag{4}$$

$$\sum_{\nu=0}^{\infty}\left(\frac{a l^{\alpha+\beta}}{\gamma}\right)^{\nu}\frac{1}{(\nu+1)!}\sum_{i=0}^{\nu}\frac{(-1)^{i}\lambda^{i}(i+1-\gamma)\cdots(i+1-\nu\gamma)}{i!}\,s^{\frac{\alpha}{\gamma}i}$$

is a locally integrable function in every domain

$$|\lambda|\leqslant\lambda_0,\quad 0\leqslant t\leqslant T$$

Multiplying the expression (3) by $\exp(-\lambda s^{\frac{\alpha}{\gamma}})$ we obtain the formula

$$\exp[-\lambda(s^{\alpha}+as^{-\beta})^{\frac{1}{\gamma}}]=\exp(-\lambda s^{\frac{\alpha}{\gamma}})\,(1+f_{\alpha\beta\gamma}(\lambda))$$

where $f_{\alpha\beta\gamma}(\lambda)$ is defined by the relation (4).

Assume $0<\frac{\alpha}{\gamma}<1$ then

$$\exp(-\lambda s^{\frac{\alpha}{\gamma}})=\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\exp(zt-z^{\frac{\alpha}{\gamma}}\lambda)\,dz\right\}\qquad \lambda>0.$$

The function in braces is continuous for $\lambda>0$, $0\leqslant t<\infty$; moreover it vanishes at $t=0$ together with all its derivatives with respect to t. This implies that also the function (1) is a parametric for $\lambda\geqslant 0$ and vanishes together with all its derivatives with respect to t for $t=0$. Consequently, for positive values λ $\exp[\lambda(s^{\alpha}+as^{-\beta})]^{\frac{1}{\gamma}}$ is not a parametric function since otherwise

$$\exp[-\lambda(s^{\alpha}+as^{-\beta})^{\frac{1}{\gamma}}]\;\exp[\lambda(s^{\alpha}+as^{-\beta})^{\frac{1}{\gamma}}]=1$$

would have to be a continuous function.

3. Approximation of the exponential function (1)

It is well known that generally it is impossible to say that one of two operators is larger than the other, just as we cannot always say this about complex numbers.

Now, we shall give the approximation of the exponential function (1) by the method which gave B. Stanković [9] [10]. At first we shall give the following definition.

D e f i n i t i o n. The operator $a \in M$ approximates the operator $b \in M$ with a factor $q \in M$ and a measure ε if

$$q^{-1}(a-b) \in \mathcal{L} \text{ and } |q^{-1}(a-b)| \leqslant \varepsilon l^{*}$$

The operator $a \in M$ approximates locally the operator $b \in M$ with a factor $q \in M$ and a measure $\varepsilon(T)$ if for every $0 \leqslant t \leqslant T$

$$q^{-1}(a-b) \in \mathcal{L} \text{ and } |q^{-1}(a-b)| \underset{T}{\leqslant} \varepsilon(T)\, l$$

(see [9] p. 186). If $q=1$ our definition gives the classical approximation in $\mathcal{L}$.

Let us suppose that we have computed the first i_0 coefficients $\binom{\frac{1}{\gamma}}{i} a^i$ $i=0, 1, \ldots, i_0$ so that $(\alpha+\beta)\, i_0 > \frac{\alpha}{\gamma}$.

As the appoximate exponential operational function of the function (1) we shall take following function

$$\exp\left[-\lambda \sum_{i=0}^{i_0} \binom{\frac{1}{\gamma}}{i} a^i s^{\frac{\alpha}{\gamma}-(\alpha+\beta) i}\right] = \tilde{x}(\lambda)$$

The difference from the exact operational function is

$$\exp\left(-\lambda \sum_{i=0}^{i_0} \binom{\frac{1}{\gamma}}{i} a^i s^{\frac{\alpha}{\gamma}-(\alpha+\beta) i}\right)\left[\exp\left(-\lambda \sum_{i \geqslant i_0+1} \binom{\frac{1}{\gamma}}{i} a^i l^{(\alpha+\beta) i-\frac{\alpha}{\gamma}}\right)-1\right] \tag{5}$$

where $(\alpha+\beta)(i_0+1) \geqslant \frac{\alpha}{\gamma}$.

One can ask the following questions

— Find the factor and measure of this approximation.

— Give an appropriate form for the approximate exponential function.

— If the approximate function is from $\mathcal{L}$ find an approximation of its suitability for a computer.

The factor of the approimation in our case as we can see from (5) is the product of elements analysed in the first part.

If such an element belongs to C or $\mathcal{L}$ it is expressed by one of the function of E. M. Wright and its properties ([9] p. 191—192) can be used to make a valuation or an approximation of it by polynomials, which is very suitable for a computer.

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* By the module $|f|$ of a function f of class $\mathcal{L}$ we shall simply understand

$$|f| = |\{f(t)\}| = \{|f(t)|\}$$

and module $|f|$ is thus again a function of class $\mathcal{L}$. If $f, g \in \mathcal{L}$ then $|f+g| \leqslant |f| + |g|$, $|fg| \leqslant |f|\,|g|$.

Because the coefficients $\binom{\frac{1}{\gamma}}{i} a^i$, $i \geqslant i_0 + 1$, are with the alternate sign it follows

$$\left| \sum_{i \geqslant i_0+1} \binom{\frac{1}{\gamma}}{i} a^i \, l^{(\alpha+\beta)\, i - \frac{\alpha}{\gamma}} \right| \underset{T}{\leqslant} \left| \binom{\frac{1}{\gamma}}{i_0 + 1} \right| a^{i_0+1} \, l^{(\alpha+\beta)(i_0+1) - \frac{\alpha}{\gamma}}$$

Now it is easy to find the measure of the approximation using the relation (5) in the following way

$$\left| \exp \left(-\lambda \sum_{i \geqslant i_0+1} \binom{\frac{1}{\gamma}}{i} a^i \, l^{(\alpha+\beta)\, i - \frac{\alpha}{\gamma}} \right) - 1 \right| =$$

$$= \left| \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left[\lambda \sum_{i \geqslant i_0+1} \binom{\frac{1}{\gamma}}{i} a^i \, l^{(\alpha+\beta)\, i - \frac{\alpha}{\gamma}} \right]^k \right| \leqslant$$

$$\leqslant \sum_{k=1}^{\infty} \frac{1}{k!} \left| \lambda \sum_{i \geqslant i_0+1} \binom{\frac{1}{\gamma}}{i} a^i \, l^{(\alpha+\beta)\, i - \frac{\alpha}{\gamma}} \right|^k$$

(6)

$$\underset{T}{\leqslant} \sum_{k=1}^{\infty} \frac{1}{k!} \left[|\lambda| \left| \binom{\frac{1}{\gamma}}{i_0 + 1} \right| a^{i_0+1} \, l^{(\alpha+\beta)(i_0+1) - \frac{\alpha}{\gamma}} \right]^k$$

$$\underset{T}{\leqslant} \exp \left(\left| \lambda \binom{\frac{1}{\gamma}}{i_0 + 1} \right| a^{i_0+1} \, l^{(\alpha+\beta)(i_0+1) - \frac{\alpha}{\gamma}} \right) - 1 =$$

$$= \left\{ t^{-1} \, \Phi \left(0, \, (\alpha+\beta)(i_0+1) - \frac{\alpha}{\gamma}; \, \left| \lambda \binom{\frac{1}{\gamma}}{i_0 + 1} \right| a^{i_0+1} \, t^{(\alpha+\beta)(i_0+1) - \frac{\alpha}{\gamma}} \right) \right\} =$$

$$= \{ t^{-1} \, \Phi (0, \, \delta; \, |\lambda| \, \nu \, t^{\delta}) \}$$

where $\delta = (\alpha+\beta)(i_0+1) - \frac{\alpha}{\gamma}$ and $\nu = \left| \binom{\frac{1}{\gamma}}{i_0 + 1} \right| a^{i_0+1}$

Example. One approximation of the exponential function

$$\exp\left[-\lambda\sqrt{s+as^{-1}}\right],\quad \lambda>0$$

is of the following form

$$\exp(-\lambda s^{1/2})\exp\left(-\lambda\sum_{k=1}^{i_0}\binom{\frac{1}{\gamma}}{k}a^k\, l^{2k-1/2}\right)=$$

$$=\{t^{-1}\Phi(0,\,-1/2;\,-\lambda t^{-1/2})\}\cdot$$

$$\prod_{k=1}^{i_0}\left[\left\{t^{-1}\Phi\left(0,\,2k-1/2;\,-\lambda\binom{1/2}{k}a^k t^{2k-1/2}\right)+1\right]\right.$$

The measure of this approximation we can find in the following way

$$\left|\exp\left[-\lambda\left(\sum_{k=i_0+1}^{\infty}\binom{1/2}{k}a^k\, l^{2k-1/2}\right)\right]-1\right|\underset{T}{\leqslant}$$

$$\underset{T}{\leqslant}\left\{t^{-1}\Phi\left(0,\,2i_0+3/2;\left|\lambda\binom{1/2}{i_0+1}\right|a^{i_0+1}\,t^{2i_0+3/2}\right)\right\}$$

From the supposition that $2i_0+3/2>0$ for $i_0=1, 2, \ldots$ it follows that the expression

$$\exp\left[-\lambda\sum_{k=i_0+1}\binom{1/2}{k}a^k\, l^{2k-1/2}\right]-1$$

is a function from $\mathscr{L}$.

Because the measure of the approximation (6) of the exponential function (1) is given by the function (1) $t^{-1}\Phi(0,\delta;|\lambda|\nu t^{\delta})$ to have an upper bound of this function we can use the Taylor series for the function

$$\{t^{-1}\Phi(0,\delta;|\lambda|\nu t^{\delta})\}=\left\{t^{-1}\sum_{k=1}^{\infty}\frac{(|\lambda|\nu t^{\delta})^k}{\Gamma(k+1)\Gamma(k\delta)}\right\}=$$

$$=\left\{\frac{t^{\delta-1}\nu|\lambda|\Gamma(\delta)}{\Gamma(\delta)}\sum_{k=1}^{\infty}\frac{(|\lambda|\nu t^{\delta})^{k-1}}{\Gamma(k+1)\Gamma(k\delta)}\right\}\underset{T}{\leqslant}$$

$$\underset{T}{\leqslant}|\lambda|\nu\Gamma(\delta)\sum_{k=0}^{\infty}\frac{(|\lambda|\nu T^{\delta})^k}{\Gamma(k+2)\Gamma[(k+1)\delta]}\,l^{\delta}=\Omega_{\delta,|\lambda|\nu}\,l^{\delta}$$

where

$$\Omega_{\delta,|\lambda|\nu}=|\lambda|\nu\Gamma(\delta)\sum_{k=0}^{\infty}\frac{(|\lambda|\nu T^{\delta})^k}{\Gamma(k+2)\Gamma[(k+1)\delta]}$$

For a bound of $\Omega_{\delta,|\lambda|\nu}$ we can consider three cases:

(i) $0<\delta<1$, (ii) $1\leqslant\delta<3/2$ and (iii) $\delta\geqslant 3/2$ (see [10] p. 82.)

If the approximate exponential function is a function of class C or $\mathscr{L}$ we can give a bound of the error.

Namely,

$$\exp\left[-\lambda\sum_{k=0}^{\infty}\binom{\frac{1}{\gamma}}{k}a^k l^{(\alpha+\beta)k-\frac{\alpha}{\gamma}}\right]-\exp\left[-\lambda\sum_{k=0}^{i_0}\binom{\frac{1}{\gamma}}{k}a^k s^{\frac{\alpha}{\gamma}-(\alpha+\beta)k}\right]\Bigg| \underset{T}{\leqslant}$$

$$\underset{T}{\leqslant}\left|\exp\left[-\lambda\sum_{k=0}^{i_0}\binom{\frac{1}{\gamma}}{k}a^k s^{\frac{\alpha}{\gamma}-(\alpha+\beta)k}\right]\right| \, |\{t^{-1}\Phi(0,\delta;|\lambda|\nu t^{\delta})\}| \underset{T}{\leqslant}$$

$$\underset{T}{\leqslant}\prod_{k=0}^{i_0}\left[\left\{t^{-1}\Phi\left(0,(\alpha+\beta)k-\frac{\alpha}{\gamma};-\lambda\binom{\frac{1}{\gamma}}{k}a^k t^{(\alpha+\beta)k-\frac{\alpha}{\gamma}}\right)\right\}+1\right]$$

$$\underset{T}{\leqslant}\prod_{k=0}^{i_0}\left[\left\{t^{-1}\Phi\left(0,(\alpha+\beta)k-\frac{\alpha}{\gamma};-\lambda\binom{\frac{1}{\gamma}}{k}a^k t^{(\alpha+\beta)k-\frac{\alpha}{\gamma}}\right)\right\}+1\right]\Omega_{\delta,|\lambda|\nu}l^{\delta}$$

$$\underset{T}{\leqslant}\sum_{k=0}^{i_0}\left|\Omega_{(\alpha+\beta)k-\alpha/\gamma,\,-\lambda\binom{1/\gamma}{k}ak}\, l^{(\alpha+\beta)k-\frac{\alpha}{\gamma}}+1\right|\Omega_{\delta,|\lambda|\nu}l^{\delta}$$

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