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A REMARK ON DECOMPOSABLE MODULES

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Introduction

Throughout, A denotes an associative ring with identity and "module" means "left, unitary A-module". An A-module M is called indecomposable if the only direct summands are 0 and M. M is called completely decomposable (abbreviated c. d.) if it is a direct sum of indecomposable injective submodules. M is irreducible if every non-zero submodule is essential in M. Then an indecomposable, injective module is irreducible [5, Proposition 2.2]. In [5], E. Matlis asked whether every direct summand of a c.d. module is c.d. The answer is positive for the following classes of c.d. modules M: (1) M injective [3, Theorem 6.4]; (2) M well-complemented (that is, the intersection of any two closed submodules of M is closed in M) [2, Theorem 3]. A result of R.E. Johnson states that non-singular modules are well-complemented (cf. [6, p. 45]). In fact, M is well-complemented if every non-zero closed submodule of M contains $Z(M)$ (cf. Lemma 1 (iii) below). The purpose of this note is to show that Matlis'conjecture holds for larger classes of c.d. modules.

Let M be an A-module, N a submodule of M. We recall that (a) N is closed in M iff N has no proper essential extension in M; (b) The closure of N in M is $Cl_M(N)=\{y\in M/(N:y)$ is essential in $A\}$ and the singular submodule of M is $Z(M)=Cl_M(0)$. M is called non-singular if $Z(M)=0$. Write $Z_2(M)\,(ClCl\,(N))$ for $Cl_M(Z(M))\,(Cl_M(N)))$. Then $ClCl\,(N)$ is the (unique) maximal essential extension of $Cl\,(N)$ in M [7, Lemma 1]. As usual, $E(N)$ will denote the injective hull of N. Obviously, $E(N)\subseteq M$ iff there exists an injective submodule Q of M such that $N\subseteq Q$.

Lemma 1. *Let M be an A-module, N a submodule of M.*

(i) *If R, Q are submodules of M such that $R\subseteq N\subseteq R\oplus Q=S$, then $Cl_S(N)= = R\oplus C_Q(N\cap Q)$;*

(ii) *If N is a direct summand of M, then $ClCl(N)=N\oplus P$, with $P\subseteq Z_2(M)$;*
(iii) *$N+Z(M)$ is essential in $Cl(N)$.*

Proof. (i) We first show $Cl_S(N)=R\oplus(Cl_S(N)\cap Q)$. Let $y\in Cl_S(N)$. Then $y=r+q$, $r\in R$, $q\in Q$ and $q=y-r\in Cl_S(N)$ implies $y\in R\oplus(Cl_S(N)\cap Q)$. Since the reverse inclusion is obvious, $Cl_S(N)=R\oplus(Cl_S(N)\cap Q)$. Now if $z\in R\oplus(Cl_S(N)\cap Q)$, $z=z_1+z_2$, $z_1\in R$, $z_2\in Cl_S(N)\cap Q$ and there exists an essential left ideal L such that $Lz_2\subseteq N\cap Q$ which implies $z_2\in Cl_Q(N\cap Q)$. Thus $R\oplus(Cl_S(N)\cap Q)\subseteq R\oplus Cl_Q(N\cap Q)$. Since $Cl_Q(N\cap Q)\subseteq Cl_S(N)$, $R\oplus Cl_Q(N\cap Q)\subseteq R\oplus(Cl_S(N)\cap Q)$ which proves (i).

(ii) If N is a direct summand of M, then N is a direct summand of $ClCl(N)$. Let $ClCl(N)=N\oplus P$. Let $0\neq y\in P$. Then $Ly\subseteq Cl(N)$ for some essential left ideal L of A. If $Ly=0$, $y\in Z(M)\subseteq Z_2(M)$. If $Ly\neq 0$, for any $k\in L$ such that $ky\neq 0$, there exists an essential left ideal J such that $Jky\subseteq N$. If $Jky\neq 0$, then $N\cap P\neq 0$, a contradiction. Therefore $Jky=0$ which implies $ky\in Z(M)$. Then $Ly\subseteq Z(M)$ which implies $y\in Z_2(M)$. This proves that $P\subseteq Z_2(M)$.

(iii) Suppose there exists a non-zero submodule P of $Cl(N)$ such that $(N+Z(M))\cap P=0$. If $0\neq y\in P\subseteq Cl(N)$, $Ly\subseteq N$ for some essential left ideal L of A which implies $Ly\subseteq N\cap P=0$. Thus $y\in Z(M)$ and $y\in(N+Z(M))\cap P=0$, a contradiction. This proves (iii).

Lemma 2. *Let M be a c.d. module. Then $Z_2(M)$ is c.d. and a direct summand of M. If N is a direct summand of M, $Z_2(N)$ is a direct summand of $Z_2(M)$.*

Proof. Let $M=(\bigoplus_{i\in I} M_i)\oplus(\bigoplus_{j\in J} M_j)$ be a direct sum of indecomposable injectives with $Z(M_i)=0$ for every $i\in I$ and $Z(M_j)\neq 0$ for every $j\in J$. Then $Z(M)=(\bigoplus_{i\in I} Z(M_i))\oplus(\bigoplus_{j\in J} Z(M_j))=\bigoplus_{j\in J} Z(M_j)$. With $S=\bigoplus_{i\in I} M_i$ and $R=\bigoplus_{j\in J} M_j$, by Lemma 1 (i), $Cl_M(R)=R\oplus Cl_S(R\cap S)=R\oplus Z(S)=R$. Since $Z(M_j)$ is essential in M_j for every $j\in J$, then $Z(M)$ is essential in R. Therefore $R\subseteq Z_2(M)\subseteq$ $\subseteq Cl_M(R)=R$ implies $R=Z_2(M)$.

If N is a direct summand of M, then by Zorn's Lemma, $N=(\bigoplus_{k\in K} N_k)\oplus W$, where each N_k is non-singular, indecomposable injective and W is an essential extension of $Z(N)$. By [7, Lemma 1], W is essential in $Z_2(N)$. Since W is a direct summand of $Z_2(N)$, then $W=Z_2(N)$ which implies $Z_2(N)$ a direct summand of $Z_2(M)$.

In [6, p. 40], G. Renault proved the equivalence of the following conditions: (i) M is well-complemented; (ii) Given a submodule P of M, P is closed in M iff for any $y\in M$, $y\notin P$ implies there exists $a\in A$ with $ay\neq 0$ and $Aay\cap P=0$. Call M a g.w.c. — module (generalised well-complemented) if M has the following property: For any injective submodule Q of M, $y\in M$, $y\notin Q$ implies there exists $a\in A$ with $ay\neq 0$ and $Aay\cap Q=0$. Clearly, M well-complemented implies M a g.w.c. — module. (Which, in turn, implies every submodule of M is g.w.c.). Next, call M a s.c.i. — module if there exists an injective submodule Q of M containing $Z(M)$. We now prove our main result which extends [3, Theorem 6.4] and [2, Theorem 3].

Theorem 3. *Let M be a c.d. module.*

(i) *If N is a direct summand of M, then $ClCl(N)$ is c.d.;*

(ii) *If N is a direct summand of M containing $Z(M)$, then N is c.d.;*

(iii) *If M is s.c.i., then every direct summand of M is c.d.;*

(iv) *If N is a g.w.c. — submodule of M and $E(C)\subseteq N$ for every cyclic submodule C of N, then N is c.d.;*

(v) *If M is g.w.c., then every direct summand of M is c.d.*

Proof. With the notation of Lemma 2, $Z_2(M)=\bigoplus_{j\in J} M_j$.

(i) By Lemma 1 (ii), $ClCl(N)=N\oplus P$, $P\subseteq Z_2(M)$. If $N=(\bigoplus_{k\in K} N_k)\oplus Z_2(N)$ as in Lemma 2, then $ClCl(N)=(\bigoplus_{k\in K} N_k)\oplus Z_2(N)\oplus P\subseteq(\bigoplus_{k\in K} N_k)+Z_2(M)$. Since $Z(\bigoplus_{k\in K} N_k)=\bigoplus_{k\in K} Z(N_k)=0$, then $(\bigoplus_{k\in K} N_k)\cap Z(M)=0$ implies $(\bigoplus_{k\in K} N_k)\cap Z_2(M)=0$ by [7, Lemma 1]. Therefore $ClCl(N)=(\bigoplus_{k\in K} N_k)\oplus Z_2(M)=(\bigoplus_{k\in K} N_k)\oplus(\bigoplus_{j\in J} M_j)$.

(ii) By Lemma 1 (iii), $N=Cl_M(N)$. Then $N=ClCl(N)$ is c.d. by (i).

(iii) Let Q be an injective module such that $Z(M)\subseteq Q\subseteq M$. Then $E(Z(M))\subseteq Q$ implies $E(Z(M))\subseteq Z_2(M)$ and by [7, Lemma 1], $Z_2(M)=E(Z(M))$ is injective. If N is a a direct summand of M, then $N=(\bigoplus_{k\in K} N_k)\oplus Z_2(N)$ with $Z_2(N)$ a direct summand of $Z_2(M)$ as in Lemma 2. Since $Z_2(M)$ is c.d. (Lemma 2), by [3, Theorem 6.4], $Z_2(N)$ is c.d. which proves N c.d.

(iv) By [4, Theorem 3], N contains an essential submodule $P=\bigoplus_{i\in I} P_i$ with P_i irreducible. If $0\neq y_i\in P_i$, $E(Ay_i)\subseteq N$. Then $N'=\bigoplus_{i\in I} E(Ay_i)$ is essential in N. We prove $N=N'$. For any $0\neq y\in N$, $E(Ay)\subseteq N$ and by [3, Proposition 6.2], $E(Ay)=\bigoplus_{s=1}^{n} R_s$ with R_s indecomposable injective. If $0\neq r\in R_s$, there exists $a\in A$ such that $0\neq ar\in N'$.

Then $ar\in\bigoplus_{j=1}^{m} E(Ay_{i_j})=Q$ (injective), $i_j\in I$. Therefore $R_s\cap Q\neq 0$ and for any non-zero submodule R of R_s, $R\cap Q\neq 0$ since R_s is irreducible. Since N is g.w.c., $R_s\subseteq Q\subseteq N'$. Then $E(Ay)\subseteq N'$ implies $y\in N'$ and hence $N=N'=\bigoplus_{i\in I} E(Ay_i)$ is c.d.

(v) Any direct summand N of M is g.w.c. and for every cyclic submodule C of N, $E(C)\subseteq N$ by [3, Lemma 6.1]. By (iv), N is c.d.

It is well-known that if A is left Noetherian, then every direct summand of any c.d. module is c.d. [5]. If every A-module M is s.c.i., A is not necessarily Noetherian (in fact, A may not satisfy the maximum condition on left annihilators). Otherwise, commutative rings whose singular modules are injective are semi-simple, Artinian by [8, Theorem 7] and [9, Theorem 2], contradicting [1, p. 161 (Remark)]. Corollary 4. *Let A be such that every A-module is either s.c.i. or g.w.c. Then every direct summand of any c.d. module c.d.*

If M is a well-complemented, c.d. module, K a closed submodule of M, then $E(F)\subseteq K$ for every finitely generated submodule F of K [2, Lemma 3]. In this direction, we have the following:

Corollary 5. *Let M be a g.w.c. module which is c.d. The following are equivalent for a submodule N:*

(i) *N is a direct summand of M;*

(ii) *N is c.d.;*

(iii) *$E(C)\subseteq N$ for any cyclic submodule C of N.*

Proof. Apply [3, Lemma 6.1], [4, Theorem 3] and Theorem 3.

Corollary 6. *Let M be a c.d. module such that $Z(M)$ is a closed, g.w.c. submodule of M. Then any direct summand of M is c.d.*

Proof. $Z(M)=Z_2(M)$ [7, Lemma 1] is c.d. and a direct summand of M by Lemma 2. If N is a direct summand of M, $N=(\bigoplus_{k\in K} N_k)\oplus Z_2(N)$ as in Lemma 2, where $Z_2(N)$ is a direct summand of $Z_2(M)$. By Corollary 5, $Z_2(N)$ is c.d. which proves N c.d.

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