PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 25 (39), 1979, pp. 95—100

ON A CLASS OF ANTI-INVERSE SEMIGROUPS

Svetozar Milić, Stojan Bogdanović

(Communicated March 9, 1978.)

In [1], anti-inverse semingroups are studied. The semigroups S is anti--inverse if for every $x \in S$ there exists $y \in S$ such that $xyx = y$ and $yxy = x$. In this paper we consider semigroups which satisfy the following condition:

$$(\forall x)\ (\exists y)\ (x^m = y^m \wedge x^m = (xy)^m \wedge x^n = x) \tag{1}$$

where $m, n \in N$. The law (1) is "generalisation" of the law in the Theorem 2.2. [1]. We denote with $\mathcal{S}_{m,n}$ the class of semigroups for which (1) holds. We denote with $\mathcal{A}$ the class of anti-inverse semigroups. Then, $\mathcal{A} = \mathcal{S}_{2,5}$ ([1], Theorem 2.2.).

L e m m a 1. *Every semigroup from* $\mathcal{S}_{1,n}$ $(n \in N)$ *is an anti-inverse semigroup.*

P r o o f. The law (1) for $m = 1$ is

$$(\forall x)\ (\exists y)\ (x = y,\quad x = xy,\quad x^n = x).$$

From which we have

$$(\forall x)\ (x = x^2).$$

Consequently, every semigroup from $\mathcal{S}_{1,n}$ is band, so it is anti-inverse [1].

L e m m a 2. *Every semigroup from* $\mathcal{S}_{2,n}$, *for* $n > 1$ *is anti-inverse semigroup.*

P r o o f. We distinguish two cases.

The case 1. $n = 2k$. In this case the law (1) is

$$(\forall x)\ (\exists y)\ (x^2 = y^2,\quad x^2 = (xy)^2,\quad x^{2k} = x)$$

from which we have

$$(\forall x)\ (\exists y)\ (x^{2k} = y^{2k},\quad x^2 = (xy)^2,\quad x^{2k} = x).$$

Then

$$(\forall x)\,(\exists y)\,(x=y,\quad x^2=(xy)^2,\quad x^{2k}=x)$$

i.e.

$$(\forall x)\,(\exists y)\,(x=y,\quad x^2=x^4,\quad x^{2k}=x).$$

It follows

$$(\forall x)\,(x^2=x^4,\quad x^{2k}=x)$$

so that

$$(\forall x)\,((x^{2k})^2=x^{2k},\quad x^{2k}=x)$$

i.e.

$$(\forall x)\,(x=x^2).$$

It implies that every semigroup from $\mathcal{S}_{2,2k}$ is anti-inverse.

The case 2. $n=2k+1$. In this case the law (1) is

$$(\forall x)\,(\exists y)\,(x^2=y^2,\quad x^2=(xy)^2,\quad x^{2k+1}=x)$$

i.e. let y be the existing element for x, so we have

(1.1) $$x^2=y^2,\quad x^2=(xy)^2,\quad x^{2k+1}=x.$$

Multiplying the second equality from the left side with x^{2k+1} we have

$$x^{2k+1}=x^{2k+1}yxy$$

so that

$$x=y^{2k+1}xy$$

(since $x^{2k}=y^{2k}$ from the first equality (1.1))
i.e.

(1.2) $$x=yxy.$$

Similarly, by multiplying the equality $y^2=(xy)^2$ with y^{2k-1} from the right side we have

$$y^{2k+1}=xyxy^{2k}$$

i.e.

(1.3) $$y=xyx^{2k+1}=xyx.$$

From (1.2) and (1.3) it follows

$$(\forall x)\,(\exists y)\,(x=yxy,\quad y=xyx).$$

Therefore, every semigroup from $\mathcal{S}_{2,2k+1}$ is an anti-inverse semigroup.

By this, the lemma is proved.

The class $\mathcal{S}_{m,1}$ $(m>1)$ is not the class of anti-inverse semigroups.

Example. The semigroup given by the table

	a	b
a	a	a
b	a	a

is from $\mathcal{S}_{m,1}$ $(m>1)$, and is not anti-inverse.

Lemma 3. *Every semigroup from $\mathcal{S}_{m,2}$ is an anti-inverse semigroup.*

Proof. For $n=2$ the law (1) is

$$(\forall x)\,(\exists y)\,(x^m=y^m,\ \ x^m=(xy)^m,\ \ x^2=x).$$

From that it follows

$$(\forall x)\,(x=x^2).$$

Therefore

$$\mathcal{S}_{m,2}\subset\mathcal{A}.$$

Lemma. 4 *Every semigroup from $\mathcal{S}_{m,m}$ $(m\in N)$ is an anti-inverse semigroup.*

Proof. For $m=n$ the law (1) is

$$(\forall x)\,(\exists y)\,(x^m=y^m,\ \ x^m=(xy)^m,\ \ x^m=x)$$

from which we have

$$(\forall x)\,(\exists y)\,(x=y,\ \ x=xy)$$

i.e.

$$(\forall x)\,(x=x^2).$$

Consequently,

$$\mathcal{S}_{m,m}\subset\mathcal{A}.$$

From these lemmas we have

Theorem. *Semigroups from the classes*

$$\mathcal{S}_{1,n},\ \ \mathcal{S}_{2,n}\,(n>1),\ \ \mathcal{S}_{m,2},\ \ \mathcal{S}_{m,m}$$

for any $m, n\in N$, are anti-inverse semigroups.

Further on, we consider the class of semigroups $\mathcal{S}_{m,n}$ for $3\leqslant m<n$.

The first step. Let $3\leqslant m<n$ and $S\in\mathcal{S}_{m,n}$. In this case we have

$$n=mq_1+r_1,\ \ \text{where}\ \ 0\leqslant r_1\leqslant m-1.$$

(i) For $r_1=0$ from (1) we have that for any $x\in S$ there exists $y\in S$ such that

$$x^m=y^m,\ \ x^m=(xy)^m,\ \ x^{mq_1}=x. \tag{1.4}$$

From the first and second equality (1.4) we get

$$x^{mq_1}=y^{mq_1},\ \ x^{mq_1}=(xy)^{mq_1}.$$

By using the third equality from (1.4) we have

$$x=y,\ \ x=xy. \tag{1.5}$$

From (1.5) we have that for every $x\in S$ is

$$x=x^2$$

so S is band.

Consequently, $\mathcal{S}_{m,mq_1}$ is a class of anti-inverse semiqroups.

(ii) For $r_1 = m-1$ we have

$$x^m = y^m, \quad x^m = (xy)^m, \quad x^{mq_1+m-1} = x. \tag{1.6}$$

From the first and second equality from (1.6) we have

$$x^{m(q_1+1)} = y^{m(q_1+1)}, \quad x^{m(q_1+1)} = (xy)^{m(q_1+1)}$$

Using the third equality from (1.6) we get

$$x^2 = y^2, \quad x^2 = (xy)^2, \quad x^{mq_1+m-1} = x \tag{1.7}$$

so $\mathcal{S}_{m,mq_1+m-1}$ is a class of anti-inverse semigroups (Theorem).

(iii) For $r_1 = 1$ we have

$$x^m = y^m, \quad x^m = (xy)^m, \quad x^{mq_1+1} = x. \tag{1.8}$$

From the first and second equality (1.8) we have

$$x^{m(q_1+1)} = y^{m(q_1+1)}, \quad x^{m(q_1+1)} = (xy)^{m(q_1+1)}.$$

Using the third equality from (1.8) which is

$$x^{m(q_1+1)-(m-1)} = x$$

we have

$$x^m = y^m, \quad x^m = (xy)^m, \quad x^{mq_1+1} = x$$

which gives relations (1.8). In this case $\mathcal{S}_{m,mq_1+1}$ is not a class of anti-inverse semigroups for any $m, q_1 \in N$.

Example. Let $m=3$, $q_1=1$. In the class $\mathcal{S}_{3,4}$ is a cyclic group of order three, which is not anti-inverse.

(iv) For $2 \leqslant r_1 \leqslant m-2$ and $m \geqslant 4$ the relation (1) becomes

$$x^m = y^m, \quad x^m = (xy)^m, \quad x^{mq_1+r_1} = x. \tag{1.9}$$

From (1.9) we have

$$x^{m(q_1+1)} = y^{m(q_1+1)}, \quad x^{m(q_1+1)} = (xy)^{m(q_1+1)}, \quad x^{m(q_1+1)-(m-r_1)} = x$$

i. e.

$$x^{m-r_1+1} = y^{m-r_1+1}, \quad x^{m-r_1+1} = (xy)^{m-r_1+1}, \quad x^n = x. \tag{1.10}$$

Consequently,

$$\mathcal{S}_{m,n} \subset \mathcal{S}_{m-r_1+1,n}.$$

Since $3 \leqslant m-r_1+1 < n$, we continue the algorithm as an the first step. We denote $m_1 = m-r_1+1$, then $3 \leqslant m_1 < n$.

The second step. Let $S \in \mathcal{S}_{m_1,n}$, then $n = m_1 q_2 + r_2$, where $0 \leqslant r_2 \leqslant m_1 - 1$, i.e. $0 \leqslant r_2 \leqslant m - r_1$.

From (1.10) we have

$$x^{m_1} = y^{m_1}, \quad x^{m_1} = (xy)^{m_1}, \quad x^{m_1 q_2 + r_2} = x. \tag{1.11}$$

For $r_2=0$, $r_2=1$, $r_2=m_1-1=m-r_1$ from (1.11) we have, respectively as in the first step (i), (ii), (iii).

Continuing the algorithm for $2\leqslant r_2\leqslant m_1-2$, $m_1\geqslant 4$, in the k-th step we have

$$m_{k-1}=m_{k-2}-r_{k-1}+1 \qquad (k\geqslant 2,\ m_0=m)$$

i.e.

$$m_{k-1}=m-\sum_{i=1}^{k-1} r_i+k-1.$$

Then $3\leqslant m_{k-1}<n$, where $m=m_0>m_1>\cdots>m_{k-1}$.

The k-th step. In this case is

$$n=m_{k-1}q_k+r_k, \quad \text{where} \quad 0\leqslant r_k\leqslant m_{k-1}-1$$

i.e.

$$0\leqslant r_k\leqslant m-\sum_{i=1}^{k-1} r_i+k-1.$$

Then from $(k-1)$-th step we get

(1.12) $$x^{m_{k-1}}=y^{m_{k-1}}, \quad x^{m_{k-1}}=(xy)^{m_{k-1}}, \quad x^{m_{k-1}q_k+r_k}=x.$$

For $r_k=0$, $r_k=1$, $r_k=m_{k-1}-1$ from (1.12) we have, respectively as in the first step (i), (ii), (iii). Further on for $2\leqslant r_k\leqslant m_{k-1}-2$, $m_{k-1}\geqslant 4$ we continue the algorithm.

For $r_k=2$ $(k=1, 2, \ldots, m-2)$ is $m_{m-3}=3$. Consequently, after $m-2$ steps, i.e. for $k=m-2$, we have

$$n=m_{m-3}q_{m-2}+r_{m-2}$$

i.e.

$$n=3q_{m-2}+r_{m-2}, \quad \text{where} \quad 0\leqslant r_{m-2}\leqslant 2.$$

For $r_{m-2}=0$, $r_{m-2}=1$, $r_{m-2}=2$ (there is no other possibility) we have, respectively, as in the first step (i), (ii), (iii).

By this, after not more than $m-2$ steps the algorithm is finished.

Example. Let $S\in\mathcal{S}_{16,67}$ i.e. S is a semigroup in which for any $x\in S$ there exists $y\in S$ such that

$$x^{16}=y^{16}, \quad x^{16}=(xy)^{16}, \quad x^{67}=x.$$

Here $m=16$, $n=67$, so $67=16\cdot 4+3$. Consequently, $r_1=3$, so $m_1=m-r_1+1=14$. Then $67=14\cdot 4+11$. Consequently, $r_2=11$ and $m_2=m_1-r_2+1=4$. From that follows $67=4\cdot 16+3$. Consequently, $r_3=3$, so $S\in\mathcal{A}$, i.e. $\mathcal{S}_{16,67}\subset\mathcal{A}$.

(v) Let $m>n>1$. Then

$$m=nq+r, \quad 0\leqslant r\leqslant n-1$$

and using (1) it follows

(1.13) $$x^{nq+r}=y^{nq+r}, \quad x^{nq+r}=(xy)^{nq+r}, \quad x^n=x.$$

From the third equality in (1.13) we have $x^{nq}=x^{q}$ and using the first two equalities from (1.13) we get

$$x^{q+r}=y^{q+r},\quad x^{q+r}=(xy)^{q+r},\quad x^{n}=x. \tag{1.14}$$

Obviously, $m=nq+r>q+r$, and if $q+r=n$ from (1.14) we get that the semigroup from the class $\mathcal{S}_{q+r,q+r}$ is anti-inverse (Theorem).

If $q+r=2$, then from (1.14) we have that the semigroup from the class $\mathcal{S}_{2,n}$ is anti inverse (Theorem).

If $3\leqslant q+r<n$, then we repeat the algorithm as in the case $m<n$.

If $q+r>n$, we continue the algorithm. After the finite number of steps we get the case similar to the relation (1.14), where $q+r\leqslant n$, and this was already discussed.

REFERENCES

[1] S. Bogdanović, S. Milić, V. Pavlović, *Anti-inverse semigroups*, Publ. Inst. Math., Belgrade, 24 (38), 1978, pp. 19—28.