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A GENERALIZATION OF THE NOTIONS OF REGULAR AND PERFECT CONVERGENCE METHOD

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Preliminaries: As it is known, the perfectness of convergence methods (f. ex., [8], Def. 6) is an important topic in the convergence theory. Here we generalize it (Def. 3) and give corresponding results (Propositions 1 and 2) as well as two more general results (Propos. 3 and 4). Also, here we present two open problems (see Remarks 2 and 3).

To formulate these topics, we need some new (more general) notions. Therefore, in the following part we give some denotations and definitions.

Denotations and definitions

Def. 1. *A convergence method P* is a functional $P=P(t)$ defined on some set $P^c \subseteq T$, where T denotes the set of all real (complex) sequences $t=(t_k)$.

For $t=(t_k) \in P^c$ we set $P(t)=P-\lim t_k$, or $P(t)=P-\lim t$, and call $P(t)$ *the P-limit* of the sequence $t=(t_k)$. Then the set P^c is said to be *the convergence domain of the method P*, or *the P-convergence domain.* In connection with these notions, if $t \in P^c$, the sequence t is said to be *P-limitable*, or *P-convergent*, to $P(t)$, while the method P is said to be *t-potent*, or *t-applicable*.

As we know, compatibility of methods P and Q on the set $T_1 \subseteq T$ means that $(\forall\, t \in P^c \cap T_1 \cap Q^c)\,(P\text{-}\lim t = Q-\lim t)$; in the case $T_1=T$, the methods P and Q are said to be compatible.

Def. 2. If $P^c \subseteq Q^c$, the method Q is said to be *P-conservative.* (Supposing that the method P is convergence in the ordinary sense (generally, when $P^c=c$), we have the ordinary notion of a conservative method Q.) In the case $P^c \subset Q^c$, the method Q is said to be *more potent* (*applicable*) than P.

When a method Q is P-conservative and compatible with P, it is said to be *P-regular* (or *P-permanent*). (Assuming P is usual convergence, we have the notion of regularity (permanence) in the ordinary sense.) If yet the method Q is more potent (applicable) than P, we say that Q is *better* than the method P.

If Q is P-conservative and P is Q-conservative too, the methods P and Q are said to be *equipotent* (or *equiapplicable*); when Q is P-regular and conversely, the methods P and Q are said to be *equivalent*.

Def. 3. A method Q is *P-perfect* in the point $t=(t_k)\in Q^c$ in relation to a class of methods K if for every P-regular and Q-conservative method $R\in K$ the relation

$$R-\lim t_k = Q-\lim t_k$$

is valid. (Obviously, taking P is ordinary convergence, we obtain the ordinary notion of perfectness.)

Remark 1. A class K in relation to which we define the notion of perfectness theoreticaly can include quite various convergence methods. But in further investigations usually it assumes that this class is consisted of methods whose nature is like to Q, i.e. that the methods Q and $R\in K$ are likewise formed. As well, we suppose that the method Q is such that Q^c is a $B_0 K$-space ([7] and [9]) and $P(t)$ is a linear (additive and continuous) functional in this space. Therefore, we need the following definition:

Def. 4. A method P is of the ZMOW[1] *type* if P^c is a $B_0 K$-space and if $P(t)$ is a linear functional in it.

Results. Now we can formulate a partial generalization of the Włodarski's known result ([8], Lemma 6):

Proposition 1. *Let P be a regular ZMOW-convergence method and K a class of methods Q such that $Q^c\supseteq P^c$ implies $Q\in K \Leftrightarrow Q|_{P^c}$ is a linear functional in the space P^c.*

Then P is perfect in a point s (in relation to the class K) if and only if $s\in\bar{c}$ (the closure $\bar{c}$ is taken in the space P^c).

The proof of this result can be realized by the scheme of the proof of the Włodarski's result. Our result is a generalization of the Włodarski's result only in the direction „$\Leftarrow$“. In the second direction, Proposition 1 gives obviously a weaker result.

Remark 2. Supposing P is a continuous method of the (X, F, x')-type ([4]), Def. 2.1) and K is the class of function methods, i.e. of (X, F, x')-or $(X, F, x')_g$-type methods ([2] and [3], respectively), the preceding assertion in the direction „$\Leftarrow$“ is obviously true. But we do not know whether it is true in the direction „$\Rightarrow$“. Now we can say only that the solution of this problem needs knowledge of the representation of a linear functional in some function and function sequence spaces.

The proposition 1 allows a generalization in sense of Def. 3. Namely, the following result is valid too.

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[1] in honour of Zeller, Mazur, Orlicz and Włodarski

P r o p o s i t i o n 2. *Let Q be a P-regular ZMOW-method and K a class of methods such that $R^c \supseteq Q^c$ implies*

$$R \in K \Leftrightarrow R|_{Q^c} \text{ is a linear functional in the space } Q^c.$$

Then Q is P-perfect in a point s (in relation to the class K) if and only if $s \in \overline{P^c}$ (the closure $\overline{P^c}$ is taken in the space Q^c).

The proof of this result can be realized by the scheme applied by L. Włodarski. As well, here we can give the commentaries analogous to those to Proposition 1.

Our following result needs the introduction of a new definition:

Def. 5. The set

$$\{t \mid t \in P^c \cap Q^c \text{ and } P-\lim t = Q-\lim t\}$$

is said to be *the compatibility domain* of the methods P and Q, and denoted by $(PQ)^c$. (Obviously, if methods P and Q are compatible $(PQ)^c$ coincides with $P^c \cap Q^c$.)

Now we can formulate a generalization of Proposition 2 in the direction „$\Leftarrow$":

P r o p o s i t i o n 3. *Let Q be a method of the ZMOW-type and K the class of methods from Proposition 2. Then $s \in \overline{(PQ)^c}$ implies that the method Q is P-perfect in the point s in relation to the class K.*

P r o o f. Suppose $R \in K$ is a P-regular and Q-conservative method and $t^{(n)} \in (PQ)^c$ $(n = 1, 2, 3, \dots)$ such that $t^{(n)} \to s$ in the space Q^c. Because $R|_{Q^c}$ is linear in Q^c and Q is a ZMOW-method, we have $R(t^{(n)}) \to R(s)$ and $Q(t^{(n)}) \to Q(s)$. Since R is P-regular and $t^{(n)} \in (PQ)^c$, $R(t^{(n)}) = P(t^{(n)})$ and $Q(t^{(n)}) = P(t^{(n)})$ $(n = 1, 2, 3, \dots)$. Therefore, $R(s) = Q(s)$, that means that Q is P-perfect in the point s in relation to the class K.

R e m a r k 3. In the case when Q is P-regular, we have $(PQ)^c = P^c$, wherefrom we obtain the mentioned part of the proposition 2 (by Proposition 3). But we do not know whether such generalization of Proposition 2 in the second direction is valid too.

Denote by Q_p^c the set of points in which a method Q is P-perfect in relation to some class of methods K and call it *the P-perfectness domain* of the method Q. By certain suppositions, the P-perfectness domain possesses some characteristics of the convergence domain of the method Q.

By the assumptions of Proposition 2, $Q_p^c = \overline{P^c}$, and so Q_p^c is a B_0 K-space too (the restriction $Q|_{Q_p^c}$ being linear in this space). In the case when $R \in K$ and $R^c \supseteq Q^c$ implies $R|_{Q^c}$ is a linear fuctional in the space Q_p^c, the inclusion $\overline{P^c} \subseteq Q_p^c$ is obviously valid (such situation we have in Remark 2).

But the P-perfectness domain is a B_0 K-space by weaker suppositions. So we have the following result:

Proposition 4. *Let P be a ZMOW-method and K a class of methods such that $R\in K$ and $R^c\supseteq Q^c$ implies that $R|_{Q^c}$ is a linear functional in the space Q^c. Then Q_p^c is a B_0K-space (under the pseudonorms of the space Q^c).*

Proof. It suffices to show that the set Q_p^c is closed in the space Q^c, i.e. that $t^{(n)}\in Q_p^c (n=1, 2, 3, \dots)$ and $t^{(n)}\to s\in Q^c$ (in Q^c) implies $s\in Q_p^c$. To prove this, suppose R is a P-regular and Q-conservative method from K. Because $Q(t)$ and $R(t)|_{Q^c}$ are linear functionals in the space Q^c, $Q(t^{(n)})\to Q(s)$ and $R(t^{(n)})\to R(s)$. Due to the perfectness of the method Q, we have $R(t^{(n)})=Q(t^{(n)}) (n=1, 2, 3, \dots)$ and therefore $R(s)=Q(s)$. So the method Q is perfect in the point s, i.e. $s\in Q_p^c$.

The class K from Proposition 4 can include all the function methods (see Remark 2). This assertion can be proved by using the following result:

Let $T_1\subseteq T$ be a B_0K-space and

$$(\forall t\in T_1)(\exists \text{ a neighbourhood } O_t \text{ of } x_0)(\forall x\in X\cap O_t)\left(\sum_{j=0}^{\infty} f_j(x)\,t_j \text{ converges}\right). \text{ Then}$$

$$(\exists \text{ a neighbourhood } O \text{ of } x_0)(\forall x\in X\cap O)(\forall t\in T_1)\left(\sum_{j=0}^{\infty} f_j(x)\,t_j \text{ converges}\right).$$

The proof of this proposition can be realized by means of the Banach's theorem on condensation of singularities ([1], Chapter I, § 4, Th. 6; see also [5] and [6]).

Varying the class K and the method P, we can obtain various subspaces of the space Q^c. Some new subspaces can be obtained by intersection by some space R^c.[1)] The obtained B_0K-space $Q^c\cap R^c$ has a subspace too. Namely, it is easy to show that the compatibility domain $(QR)^c$ of the methods Q and R is a B_0K-space with the pseudonorms of the space $Q^c\cap R^c$.

In the end we give some remark in connection with the considered topics. So if a method Q is P-perfect in relation to some class K, it is P-perfect in relation to every class $K_1\subseteq K$ too. As we have seen, $s\in\overline{P^c}$ guarantees P-perfectness in the point s in relation to the most extensive class K, and conversely. In connection with this fact, we arrive at the question about a minimal class K such that P-perfection in relation to this class K implies $s\in\overline{P^c}$. This question is of a special interest in case of more concrete methods Q and P. Also, it arises the question of existence of methods P, Q and a class K such that

$$P^c\subset\overline{P^c}\subset Q_p^c\subset Q^c,$$

as some other similar questions.

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1) the intersection of two B_0K-spaces is a B_0K-space too ([9], Th. 4. 7);

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