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ON SOLVING A SYSTEM OF BALANCED FUNCTIONAL EQUATIONS ON QUASIGROUPS II

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Continuing the first part of this work, we generalise results of [17] to the case where the given system of equations is not necessarily irreducible.

The results we obtain are generalisations of numerous results on functional equations on quasigroups, especially from [2], [4], [5], [6], [8], [9].

For undefined notions and notation see [7] and [1].

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Lemma *Any system* Γ *of general, balanced functional equations on quasigroups of extended equality type is equivalent to the following system: first part* $\Gamma^{=}$ *of the system is composed of equations explicitely expressing reducible operations of* Γ *in terms of its irreducible retracts. Second part* Γ^{*} *of the system we obtain from* Γ *substituting all reducible operations by their irreducible retracts, making use of equations from* $\Gamma^{=} \cdot \Gamma^{*}$ *is irreducible system of general balanced functional equations on quasigroups, of extended equality type.*

Proof: The proof is established by expressing a reducible operation of maximal arity in terms of its retracts of lesser arity (not necessarily irreducible).

Γ is a consistent system. The proof is similar to the proof of consistency of irreducible system ([7], p. 119.).

Let a solution of Γ (on a set S) be given. We interpret operational symbols from Γ as respective quasigroups from the given solution. The obtained system we denote by $\Gamma[S]$.

Let A be a reducible k-ary operation such that there are not any k'-ary operations for $k'>k$. At least one of operations from $A^{\sim}$, say B, is locally reducible. Choose the set of variables $\{y_1, \ldots, y_k\}=\{x_i \mid i\in M\}$ in such a way that $y_i\in f_B(i)$. Some M-consequence (for example of $t_1=t_2$) is as follows:

$$(1) \qquad \langle \cdot, B\rangle B(\langle B, y_1\rangle y_1, \ldots, \langle B, y_k\rangle y_k)=\langle \cdot, (C)\rangle C(T_1, \ldots, T_m)$$

where at least two variables occur in at least one of $T_1, \dots, T_m$; $(C) = \inf\{x_i \mid i \in M\}$ (inf from semilattice corresponded to t_2) and $C = (C)_N$ for $N = \bigcup_{i=1}^{k} \{\arg((C), y_i)\}$.

Let $\{y_{r_1}, \dots, y_{r_p}\}$ be the set of all variables occurring in T_1 $(p>1)$, $K = \{1, \dots, k\}$, $L = \{r_1, \dots, r_p\}$ and $L' = (K \setminus L) \cup \{r_1\}$. (2) and (3) are L- and L'-consequences of (1) $(q = k - p + 1)$:

$$\text{(2)} \qquad \langle \cdot, (C) \rangle C_{\{1\}} T_1 = \langle \cdot, B \rangle B_L (\langle B, y_{r_1} \rangle y_{r_1}, \dots, \langle B, y_{r_p} \rangle y_{r_p})$$

$$\text{(3)} \qquad \langle \cdot, (C) \rangle C (\langle (C), y_{r_1} \rangle y_{r_1}, T_2, \dots, T_m) = \langle \cdot, B \rangle B_{L'} (\langle B, y_{s_1} \rangle y_{s_1}, \dots, \langle B, y_{s_q} \rangle y_{s_q})$$

From (1), (2), (3) and $\{r_1\}$-consequence of (1):

$$\langle \cdot, B \rangle B_{\{r_1\}} \langle B, y_{r_1} \rangle y_{r_1} = \langle \cdot, (C) \rangle C_{\{1\}} \langle (C), y_{r_1} \rangle y_{r_1}$$

we get:

$$\langle \cdot, B \rangle B (\langle B, y_1 \rangle y_1, \dots, \langle B, y_k \rangle y_k) = \langle \cdot, (C) \rangle C (T_1, \dots, T_m) =$$
$$\langle \cdot, B \rangle B_{L'} (\langle B, y_{s_1} \rangle y_{s_1}, \dots, \langle B, y_{r_1} \rangle \langle (C), y_{r_1} \rangle^{-1} T_1, \dots$$
$$\dots, \langle B, y_{s_q} \rangle y_{s_q}) = \langle \cdot, B \rangle B_{L'} (\langle B, y_{s_1} \rangle y_{s_1}, \dots$$
$$\dots, \langle B, y_{r_1} \rangle \langle (C), y_{r_1} \rangle^{-1} \overset{-1}{C_{\{1\}}} \langle \cdot, (C) \rangle^{-1} \langle \cdot, B \rangle B_L (\langle B, y_{r_1} \rangle y_{r_1}, \dots$$
$$\dots, \langle B, y_{r_p} \rangle y_{r_p}), \dots \langle B, y_{s_q} \rangle y_{s_q}) = \langle \cdot, B \rangle B_{L'} (\langle B, y_{s_1} \rangle y_{s_1}, \dots$$
$$\dots, \overset{-1}{B_{\{r_1\}}} B_L (\langle B, y_{r_1} \rangle y_{r_1}, \dots, \langle B, y_{r_p} \rangle y_{r_p}), \dots, \langle B, y_{s_q} \rangle y_{s_q}).$$

It is also $L \cap L' = \{r_1\}$ so $\overset{-1}{B_{\{r_1\}}} = \overset{-1}{B_{L \cap L'}}$.

Substituting z_i for $\langle B, y_i \rangle y_i$ we get:

$$\text{(4)} \qquad B(z_1, \dots, z_k) = B_{L'} (z_{s_1}, \dots, \overset{-1}{B_{L' \cap L}} B_L (z_{r_1}, \dots, z_{r_p}), \dots, z_{s_q})$$

We can conclude that B is expressible through its retracts of lesser arity and that arguments of any of such a retract $B_{L'}$ are either a variable or a term $\overset{-1}{B_{L' \cap L}} B_L(\cdots)$, where $B_{L' \cap L}$ is unary. The term expressing B is not uniquely determined, depending on the choice of variables $y_1, \dots, y_k$.

Equation (4) constitutes the system $\Gamma_1^{=}[S]$. If retracts $B_{L'}$ and B_L are irreducible than (4) is in $\Gamma^{=}[S]$. Otherwise, after a finitely many steps, we can express reducible ones through their retracts (and also retracts of B) of lesser arity, and so on, while irreducible retracts of B are obtained. Then equation (5), obtained from (4), expressing B in terms of its irreducible retracts, is in $\Gamma^{=}[S]$.

$$\text{(5)} \qquad B(w_1, \dots, w_k) = T_B(B_L, B' B_M, \dots, B^{(m)} B_N, w_1, \dots, w_k)$$

System $\Gamma_1^*[S]$ we obtain from $\Gamma[S]$ substituting

$$B_{L'}(t_{s_1}, \dots, \overset{-1}{B_{L' \cap L}}(t_{r_1}, \dots, t_{r_p}), \dots, t_{s_q}) \text{ for } B(t_1, \dots, t_k).$$

Systems $\Gamma[S]$ and $\Gamma_1^=[S]$, $\Gamma_1^*[S]$ are equivalent.

Repeating the procedure we get sequence $(\Gamma_j^=[S], \Gamma_j^*[S])_{j=1,2,\ldots,l}$ of equivalent systems. The last member of this finite sequence must be the system of desired form.

The procedure and formulas did not depend on S. So $\Gamma_l^=, \Gamma_l^*$ is equivalent to Γ where "retracts" as B_2 are new functional letters.

Theorem *Let Γ be a general system of balanced functional equations on quasigroups, of extended equality type. Let Γ^-, Γ^* be system obtained from Γ as in Lemma, and $\Gamma^=$ be system of formulas (5) for any (reducible) operation B from Γ. The general solution of Γ is given by:*

$$(*)\qquad A(x_1,\ldots,x_k)=\langle\cdot, A\rangle^{-1}T_A(Q_{A_{\tilde{L}}}^{\pi_{AL}}, Q_{A_{\tilde{M}}}^{\pi_{AM}},\ldots$$
$$\ldots, Q_{A_{\tilde{N}}}^{\pi_{AN}}, \langle\cdot, A\rangle A_{\{1\}}x_1,\ldots,\langle\cdot, A\rangle A_{\{k\}}x_k)$$

where (for all operations B, C occurring in Γ^*):

(1) *S is any nonempty set*

(2) *$Q_{B^\sim}=Q_{C^\sim}$ for $B\sim C$ ($B\sim C$ iff diisotopy of B and C is a consequence of Γ)*

(3) *all permutations $\pi_B(B\sim C)$ on $\{1,\ldots,k\}$ are uniquely determined if one of them is given.*

(4) *$Q_{B^\sim}$ is any k-ary loop on S iff in every term of Γ^* occurs exactly one operation from $B^\sim$.*

(5) *$Q_{B^\sim}$ is any binary group on S iff in some term of Γ^* occurs at least two operations from $B^\sim$.*

(6) *$Q_{B^\sim}$ is any binary abelian group on S iff no exchange of some operations occurring in $\Gamma^*(B^\sim)$ with dual operations, transforms $\Gamma^*(B^\sim)$ in a system of the first kind.*

$$(7)\qquad \ldots, A_{\{1\}},\ldots, A_{\{k\}},\ldots$$

are permutations on S for which the following equations hold.

$$(7_i)\qquad \langle\cdot, x_i\rangle_1=\cdots=\langle\cdot, x_i\rangle_{\bar{n}}\qquad (i=1,\ldots,n)$$

where n is the number of variables occurring in Γ and $\langle\cdot, x_i\rangle_r$ is a composition $\langle\cdot, x_i\rangle$ in the term t_r.

Proof: For any k-ary quasigroup A, quasigroup Q defined as

$$Q(y_1,\ldots,y_k)=PA^{\pi-1}(A_{\{1\}}^{-1}P^{-1}y_1,\ldots,A_{\{k\}}^{-1}P^{-1}y_k)$$

where $A_{\{i\}}x=A(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_k)$, must be a loop with $PA(a_1,\ldots,a_k)$ as unit.

(4) then follows from (4) of Theorem of [7] (p. 125), where we can also change the word quasigroup by the word loop.

Let term T_A (from (5)) be of the following form:

$$T_A(\cdots)=A_L(\ldots,\ x_j,\ \ldots,\ A_{L\cap M}^{-1}A_M(\cdots),\cdots).$$

$$A(x_1,\ \ldots,\ x_k)=\langle\cdot,\ A\rangle^{-1}Q_{A\tilde{L}}^{\pi AL}(\ldots,\ \langle\cdot,\ A\rangle A_{\{j\}}x_j,\ \ldots$$

$$\ldots,\ \langle\cdot,\ A\rangle A_{\{j'\}}A_{L\cap M}^{-1}A_M(\cdots),\cdots)=$$

$$=\langle\cdot,\ A\rangle^{-1}Q_{A\tilde{L}}^{\pi AL}(\ldots,\ \langle\cdot,\ A\rangle A_{\{j\}}x_j,\ldots$$

$$\ldots,\ \langle\cdot,A\rangle\langle\cdot,\ A\rangle^{-1}Q_{A\tilde{M}}^{\pi AM}(\cdots),\ \ldots)=$$

$$=\langle\cdot,\ A\rangle^{-1}Q_{A\tilde{L}}^{\pi AL}(\ldots,\ \langle\cdot,A\rangle A_{\{j\}}x_j,\ \ldots,\ Q_{A\tilde{M}}^{\pi AM}(\cdots),\ \ldots)$$

thus proving (*).

Other propositions follow from lemma and corresponding propositions of theorem of [7].

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In all examples, for the sake of simplicity, we write A_{123} instead of $A_{\{1,2,3\}}$ and analogously for other cases.

Example 1. Let following system Γ be given:

$$A(x_1,\ B(Y(x_3,\ x_7,\ Z(x_4,\ x_5,\ x_6),\ x_2),\ x_8))=$$

$$=W(x_1,\ x_8,\ I(V(K(x_4,\ x_5),\ x_6,\ x_3),\ x_7,\ x_2))=$$

$$=M(x_8,\ U(x_1,\ x_7,\ x_2,\ x_4,\ R(S(x_5,\ x_3),\ x_6))),$$

i.e. $t_1=t_2=t_3$.

Reducible operations are U, V, W, Y, Z. We express U in terms of its irreducible retracts.

$\{1,\ 2,\ 3,\ 4,\ 7\}$ — consequence of $t_3=t_2$ is

$$(0)\qquad M_2U(x_1,\ x_7,\ x_2,\ x_4,\ R_1S_2x_3)=W_{13}(x_1,\ I(V_{13}(K_1x_4,\ x_3),\ x_7,\ x_2))$$

From $\{1,\ 2\}$–, $\{2,\ 3,\ 7\}$– and $\{3,\ 4\}$–consequences of (0) (or $t_3=t_2$) we get:

$$W_{13}(x,\ y)=M_2U_{13}(x,\ I_3^{-1}y)$$

$$I(x,\ y,\ z)=W_3^{-1}M_2U_{235}(y,\ z,\ R_1S_2V_3^{-1}x)$$

$$V_{13}(x,\ y)=I_1^{-1}W_2^{-1}M_2U_{45}(K_1^{-1}x,\ R_1S_2y)$$

and we conclude, making use of:

$$M_2U_3=W_3I_3$$

$$M_2U_5R_1S_2=W_3I_1V_3$$

that:

(U) $$U(x, y, z, u, v) = U_{13}(x, U_3^{-1} U_{235}(y, z, U_5^{-1} U_{45}(u, v)))$$

Analogously we get:

$$V(x, y, z) = V_{12}(x, V_2^{-1} V_{23}(y, z))$$

$$W(x, y, z) = W_{12}(W_1^{-1} W_{13}(x, z), y)$$

$$Y(x, y, z, u) = Y_{124}(Y_1^{-1} Y_{13}(x, z), y, u)$$

$$Z(x, y, z) = Z_{13}(Z_1^{-1} Z_{12}(x, y), z)$$

These equations as well as (U) constitute system $\Gamma^{=}$. System Γ^* is:

$$A(x_1, B(Y_{124}(Y_1^{-1} Y_{13}(x_3, Z_{13}(Z_1^{-1} Z_{12}(x_4, x_5), x_6)), x_7, x_2), x_8)) =$$

$$= W_{12}(W_1^{-1} W_{13}(x_1, I(V_{12}(K(x_4, x_5), V_2^{-1} V_{23}(x_6, x_3)), x_7, x_2)), x_8) =$$

$$= M(x_8, U_{13}(x_1, U_3^{-1} U_{235}(x_7, x_2, U_5^{-1} U_{45}(x_4, R(S(x_5, x_3), x_6)))))$$

Next, we define some new quasigroups:

$$C(x, y, z) = Y_{124}(y, z, x)$$

$$D(x, y) = Y_1^{-1} Y_{13}(x, y)$$

$$E(x, y) = Z_{13}(x, y)$$

$$F(x, y) = Z_1^{-1} Z_{12}(x, y)$$

$$G(x, y) = W_{12}(x, y)$$

$$H(x, y) = W_1^{-1} W_{13}(y, x)$$

$$J(x, y) = V_{12}(x, y)$$

$$L(x, y) = V_2^{-1} V_{23}(y, x)$$

$$N(x, y) = U_{13}(x, y)$$

$$P(x, y, z) = U_3^{-1} U_{235}(x, y, z)$$

$$Q(x, y) = U_5^{-1} U_{45}(x, y)$$

Using these definitions Γ^* becomes Γ of example 1 from [7], i.e.:

$$A(x_1, B(C(x_2, D(x_3, E(F(x_4, x_5), x_6)), x_7), x_8)) =$$

$$= G(H(I(J(K(x_4, x_5), L(x_3, x_6)), x_7\ x_2), x_1), x_8) =$$

$$= M(x_8, N(x_1, P(x_7, x_2, Q(x_4, R(S(x_5, x_3), x_6)))))$$

Knowing general solution of Γ from [7] we have general solution of Γ:

$A(x, y) = A_1 x * A_2 y$

$B(x, y) = A_2^{-1}(A_2 B_1 x * A_2 B_2 y)$

$Y(x, y, z, u) = (A_2 B_1)^{-1} T(A_2 B_1 Y_4 u, A_2 B_1 Y_1 x + A_2 B_2 Y_3 z, A_2 B_1 Y_2 y)$

$Z(x, y, z) = (A_2 B_1 Y_3)^{-1}((A_2 B_1 Y_3 Z_1 x + A_2 B_1 Y_3 Z_2 y) + A_2 B_1 Y_3 Z_3 z)$

$W(x, y, z) = (W_1 x * W_3 z) * W_2 y$

$I(x, y, z) = W_3^{-1} T(W_3 I_3 z, W_3 I_1 x, W_3 I_2 y)$

$V(x, y, z) = (W_3 I_1)^{-1}(W_3 I_1 V_1 x + (W_3 I_1 V_3 z + W_3 I_1 V_2 y))$

$K(x, y) = (W_3 I_1 V_1)^{-1}(W_3 I_1 V_1 K_1 x + W_3 I_1 V_1 K_2 y)$

$M(x, y) = M_2 y * M_1 x$

$U(x, y, z, u, v) = M_2^{-1}(M_2 U_1 x * T(M_2 U_3 z, M_2 U_4 u + M_2 U_5 v, M_2 U_2 y))$

$R(x, y) = (M_2 U_5)^{-1}(M_2 U_5 R_1 x + M_2 U_5 R_2 y)$

$S(x, y) = (M_2 U_5 R_1)^{-1}(M_2 U_5 R_1 S_1 x + M_2 U_5 R_1 S_2 y)$

where $A_1, \ldots, S_2$ are arbitrary permutations for which the following equations hold:

$$A_1 = W_1 = M_2 U_1$$

$$A_2 B_1 Y_4 = W_3 I_3 = M_2 U_3$$

$$A_2 B_1 Y_1 = W_3 I_1 V_3 = M_2 U_5 R_1 S_2$$

$$A_2 B_1 Y_3 Z_1 = W_3 I_1 V_1 K_1 = M_2 U_4$$

$$A_2 B_1 Y_3 Z_2 = W_3 I_1 V_1 K_2 = M_2 U_5 R_1 S_1$$

$$A_2 B_1 Y_3 Z_3 = W_3 I_1 V_2 = M_2 U_5 R_2$$

$$A_2 B_1 Z_2 = W_3 I_2 = M_2 U_2$$

$$A_2 B_2 = W_2 = M_1$$

$*$ is an arbitrary group, T an arbitrary 3-loop and $+$ an arbitrary abelian group. All of them are defined on a given nonempty set X.

Example 2. Let following system Γ be given:

$$A(x_1, B(x_2, x_3)) = C(x_2, D(x_3, x_1)) = E(x_3, F(x_1, x_2))$$

System Γ is irreducible. There exists a group $*$ diisotopic to all of A, B, C, D, E, F and $*$ must be abelian because Γ is a system of the second kind. If

$$A(x, y) = A_1 x * A_2 y$$

$$B(x, y) = A_2^{-1}(A_2 B_1 x * A_2 B_2 y)$$

$$C(x, y) = C_2 y * C_1 x$$

$$D(x, y) = C_2^{-1}(C_2 D_2 y * C_2 D_1 x)$$

$$E(x, y) = E_2 y * E_1 x$$

$$F(x, y) = E_2^{-1}(E_2 F_1 x * E_2 F_2 y)$$

then commutativity follows from $\{2, 3\}$ – consequence of Γ. Analogously for any other choice of formulas of general solution.

Example 3.[1] Let $t_k = A^{(k)}(x_1^{k-1}, A^{(n+k)}(x_k^{n+k-1}), x_{n+k}^{2n-1})$, $k = 1, \ldots, n$. $t_1 = \cdots = t_n$ is the system of the functional equations of general associativity for n-ary quasigroups.

Lemma 3.1. *All the retracts of the operations $A^{(m)}$ $(m = 1, \ldots 2n)$ are reducible.*

Proof: It is sufficient to prove that all ternary retracts of operations $A^{(m)}$ $(m = 1, \ldots, 2n)$ are reducible.

If, according to $\{p, q, r\}$-consequence of an equation φ of the system, $A_{ijk}^{(m)}$ is a superposition of some binary retracts, and if M contains p, q, r and indices of variables from all arguments of $A^{(m)}$, then, according to M-consequence of φ, $A^{(m)}$ is a superpostion of some retracts and consequently reducible.

All ternary retracts of $A^{(m)}$ $(m = 2, \ldots, n, n+2, \ldots, 2n)$ are either locally reducible in equation $t_1 = t_m$ (i. e. $t_1 = t_{m-n}$ for $m > n$) or isotopic to a ternary retract of $A^{(1)}$ or $A^{(n+1)}$.

Reducibility of ternary retracts of $A^{(1)}$ and $A^{(n+1)}$ follows from:

$$A_{1jk}^{(1)}(x_1, x_{n+j-1}, x_{n+k-1}) = A_{1n}^{(n)}(x_1, A_{jk}^{(2n)}(x_{n+j-1}, x_{n+k-1}))$$

$$A_{ijk}^{(1)}(x_{n+i-1}, x_{n+j-1}, x_{n+k-1}) = A_{jk}^{(j)}(A_{n+i-jn}^{(n+j)}(x_{n+i-1}, x_{n+j-1}), x_{n+k-1}) \quad \text{for } i \neq 1$$

$$A_{ijk}^{(n+1)}(x_i, x_j, x_k) = A_{ij}^{(j)}(x_i, A_{1j+k-1}^{(n+j)}(x_j, x_k))$$

where $1 \leqslant i < j < k < n$.

Lemma 3.2. *All binary retracts of all the operations $A^{(m)}$ $(m = 1, \ldots 2n)$ are isotopic.*

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[1] Communicated December 24, 1976.

Proof: Any binary retract has two of $x_1, \dots, x_{2n-1}$ for arguments. All binary retracts with x_k and x_m $(1 \leqslant k < m < 2n)$ as arguments, are isotopic.

Let $1 \leqslant i < n$ and $i < j < n+i$. Then there is a binary retract with x_i, x_j and x_i, x_{j+1} as arguments. For example $A^{(i+1)}_{i\,i+1}$ is such one because t_{i+1} becomes $A^{(i+1)}_{i\,i+1}(x_i, A^{(n+i+1)}_{j-i}x_j)$ i.e. $A^{(i+1)}_{i\,i+1}(x_i, A^{(n+i+1)}_{j-i+1}x_{j+1})$ when $x_r = a_r$ for $r = 1, \dots$ $\dots, 2n-1$; $r \neq i, j$ $(r \neq i, j+1)$. Similarly for $i < n \leqslant j < 2n-1$, t_n becomes $A^{(n)}_{in}(x_i, A^{(2n)}_{j-n+1}x_j)$ and $A^{(n)}_{in}(x_i, A^{(2n)}_{j-n+2}x_{j+1})$.

For $j-n < i < j$ and $n < j \leqslant 2n-1$ there is a binary retract with x_i, x_j and x_{i-1}, x_j as arguments. For $x_r = a_r$ $(r = 1, \dots, 2n-1;\ r \neq i, j$ or $r \neq i-1, j)$, t_{j-n} becomes $A^{(j-n)}_{j-n\,j-n+1}(A^{(j)}_{n+i-j+1}x_i, x_j)$ i.e. $A^{(j-n)}_{j-n\,j-n+1}(A^{(j)}_{n+i-j}x_{i-1}, x_j)$. So we proved that any two binary retracts are isotopic.

The system is of the first kind and there is more than n binary retracts so all of them are isotopic to some group $*$. The general solution is:

$$A^{(m)}(x_1^n) = \prod_{k=1}^{n} A^{(m)}_k x_k \quad \text{for} \quad m = 1, \dots, n$$

$$A^{(m)}(x_1^n) = [A^{(m-n)}_{m-n}]^{-1} \prod_{k=1}^{n} A^{(m-n)}_{m-n} A^{(m)}_k x_k \quad \text{for} \quad m = n+1, \dots, 2n$$

where $\prod_{k=1}^{n} x_k = x_1 * \cdots * x_n$ and $*$ is an arbitrary group and $A^{(m)}_k$ $(m = 1, \dots, 2n;$ $k = 1, \dots, n)$ arbitrary permutations for which the following equations hold:

$$A^{(1)}_1 A^{(n+1)}_1 = A^{(2)}_1 = \cdots = A^{(n)}_1$$

$$A^{(1)}_1 A^{(n+1)}_2 = A^{(2)}_2 A^{(n+2)}_1 = A^{(3)}_2 = \cdots = A^{(n)}_2$$

$\dots$

$$A^{(1)}_1 A^{(n+1)}_{n-1} = A^{(2)}_{(2)} A^{(n+2)}_{n-2} = \cdots = A^{(n-1)}_{n-1} A^{(2n-1)}_1 = A^{(n)}_{n-1}$$

$$A^{(1)}_1 A^{(n+1)}_n = A^{(2)}_2 A^{(n+2)}_{n-1} = \cdots = A^{(n)}_n A^{(2n)}_1$$

$$A^{(1)}_2 = A^{(2)}_2 A^{(n+2)}_n = \cdots = A^{(n)}_n A^{(2n)}_2$$

$\dots$

$$A^{(1)}_{n-1} = A^{(2)}_{n-1} = \cdots = A^{(n-2)}_{n-1} = A^{(n-1)}_{n-1} A^{(2n-1)}_n = A^{(n)}_n A^{(2n)}_{n-1}$$

$$A^{(1)}_n = A^{(2)}_n = \cdots = A^{(n-1)}_n = A^{(n)}_n A^{(2n)}_n.$$

For any group$*$ $G(x_1, \dots, x_n) = x_1 * \cdots * x_n$ is an n-ary group with unit. Using the fact that any n-ary group $G(x_1, \dots, x_n)$ with unit can be presented in the form $x_1 * \cdots * x_n$ for some binary group $*$, we obtain:

Corollary. *All quasigroups $A^{(m)}$ $(m = 1, \dots, 2n)$ satisfying the system of functional equations of general associativity for n-ary quasigroups, are isotopic to an n-ary group with unit.*

Thus we obtain the main theorems (T1) and (P.1) of [6].

Example 4. If system Γ is an equation $t_1 = t_2$ where terms t_1, t_2 are of height 2, i.e. of the form $A(u_1^m)$ where $u_i (i = 1, \ldots, m)$ are either a variable x_l or a term $B_i(x_{i_1}^{i_r})$, we obtain results of B. Alimpić [2].

In this article, as well as in [7] I was greatly inspired by [2].

Example 5. If system Γ consists of only one equation, we obtain results of Belousov and Livšic on solutions of balanced functional equations of the second kind for quasigroups of arbitrary arity (see [4], [5], [8] and [9]).

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