PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 25 (39), 1979, pp. 63—69

ON NEARLY PARACOMPACT SPACES

Ilija Kovačević,

(Received August 8, 1978)

In a recent paper [5], M. K. Singal and S. P. Arya have introduced a new class of topological spaces called nearly paracompact spaces. These are characterised by the following property: "Every regular open covering of the space admits a locally finite open refinement". The class of nearly paracompact spaces contains the class of paracompact spaces. In another paper [3], another class of spaces called almost paracompact spaces has been introduced. A space is said to be almost paracompact iff for every open cover of the space there exists a locally finite family of open sets which refines it and the closures of whose members cover the space. The class of almost paracompact spaces contains the class of nearly paracompact spaces.

In the present paper, it is proposed to present some further results on nearly paracompact spaces. Notation is standard except that $\alpha(A)$ will be used to denote interior of the closure of A. The topology τ^* is the semi regularisation of τ and has as base the regularly open sets from τ.

1. Definition and characterisations

Definition 1.1. A space X is said to be almost normal iff for every pair of disjoint sets A and B, one of which is closed and the other is regularly closed, there exist disjoint open sets U and V such that $A \subset U$, $B \subset V$ [5],

Lemma. 1.1. *Let X be an almost normal space. If $\mathcal{U}=\{U_\alpha : \alpha \in I\}$ is any regular open locally finite cover of the space X, there is regular open locally finite refinement $\mathcal{V}=\{V_\alpha : \alpha \in I\}$ such that $\overline{V}_\alpha \subset U_\alpha$ for each $\alpha \in I$.*

Proof. Similar to the proof of the corresponding result for normal space.

Theorem 1.1. *For an almost normal space X, the following are equivalent.*

(a) *X is nearly paracompact,*

(b) *every regular open cover of X has a regular open locally finite refinement,*

(c) *every regular open cover of X has a locally finite, closed refinement,*

(d) *ever regular open cover of X has a locally finite, regularly closed refinement.*

Proof. $(a)\rightarrow(b)$. Let $\mathcal{U}$ be any regular open cover of X. Then there exists a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$. Consider the family $\mathcal{W}:=\{\alpha(V):V\in\mathcal{V}\}$. Then $\mathcal{W}$ is a locally finite regular open refinement of $\mathcal{U}$.

$(b)\rightarrow(c)$. It follows easily from lemma 1.1.

$(c)\rightarrow(d)$. It follows easily from lemma 1.1. in [5].

$(d)\rightarrow(a)$. Let $\mathcal{U}$ be any regular open cover of X There exists a locally finite regular closed refinement $\mathcal{V}$ of $\mathcal{U}$. Since the family $\mathcal{V}$ is locally finite, for each $x\in X$ there exists an open set 0_x such that $x\in 0_x$ and 0_x intersects finitely many members of $\mathcal{V}$. Then $\alpha(0_x)$ is a regular open set, such that $x\in\alpha(0_x)$, which intersects finitely many members of $\mathcal{V}$.

Now, consider the regular open covering $\mathcal{G}=\{\alpha(0_x):x\in X\}$ of X. Then there exists a locally finite, regular closed refinement $\mathcal{G}'$ of $\mathcal{G}$.
Let

$$V^*=X\setminus\bigcup\{G:G\in\mathcal{G}',\ G\cap V=\varnothing\}.$$

Clearly, V^* is open and contains V. Moreover, $G\cap V^*\neq\varnothing\Leftrightarrow G\cap V\neq\varnothing$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, for each $V\in\mathcal{V}$, there exists a U_V such that $V\subset U_V$. Let

$$\mathcal{W}=\{V^*\cap U_V:V\in\mathcal{V}\}.$$

$\mathcal{W}$ is an open, locally finite refinement of $\mathcal{U}$, thus X is nearly paracompact.

Definition 1.2. ([5]). Let $\mathcal{A}$ be a family of subsets of a space X. The star of a point $x\in X$, $St(x,\mathcal{A})$, in $\mathcal{A}$ defined to be the union of all members of $\mathcal{A}$ which contain x. A family $\mathcal{A}$ of subsets of a space X is said to be a star refinement of another family $\mathcal{B}$ of subsets of X iff the family of all stars of points of X in $\mathcal{A}$ forms a covering of X which refines $\mathcal{B}$.

Theorem 1.2. *Let X be an almost normal nearly paracompact space. Then. every regular open covering of X has a regular open star refinement.*

Proof. Let $\mathcal{U}=\{U\}$ be any regular open covering of X. Since X is almost normal and nearly paracompact, there exists, by theorem 1.1, a locally finite, regular closed refinement $\mathcal{V}$ of $\mathcal{U}$. For each $V\in\mathcal{V}$, there exists a $U_V\in\mathcal{U}$ such that $V\subset U_V$. For each point $x\in X$, let

$$M_x=\bigcap\{U_V:x\in V\in\mathcal{V}\}\cap\{\bigcap\{X\setminus V:x\notin V\in\mathcal{V}\}\}.$$

Since $\mathcal{V}$ is a locally finite family of regularly closed sets and every subfamily of a locally finite family is locally finite, it follows that $\cup\{V:x\notin V\in\mathcal{V}\}$ is regularly closed ([5]). Therefore

$$\cap\{X\setminus V:x\notin V\in\mathcal{V}\}=X\setminus\cup\{V:x\notin V\in\mathcal{V}\}$$

is regularly open Also, $\mathcal{V}$ being locally finite, each point of X can belong to at most a finite number of $\mathcal{V}$ and therefore the intersection of finitely many regularly open sets being regularly open, it follows that $\cap\{U_V:x\in V\in\mathcal{V}\}$ is regularly open. Thus each M_x is regularly open set. Consider, now, the family $\mathcal{M}=\{M_x:x\in X\}$. Let $x\in X$. Then either $x\in V\in\mathcal{V}$ or $x\notin V\in\mathcal{V}$. Thus in any case $x\in M_x$ and therefore $\mathcal{M}$ is a regular open covering of X. We shall prove that $\mathcal{M}$ is a star refinement of $\mathcal{U}$. Let $m\in X$ be fixed. Since $\mathcal{V}$ covers X, there exists $V\in\mathcal{V}$ such that $m\in V$. We shall show that $St(m, \mathcal{M})\subset U_V$. Let M_x be any member of $\mathcal{M}$ which contains m. If $x\notin V$, then $m\in M_x\subset X\setminus V$ by the definition of M_x. But this is a contradiction. Therefore x must belong to V, and then $M_x\subset U_V$. Thus every member of $\mathcal{M}$ contains m, is contained in U_V, and therefore $St(m, \mathcal{M})\subset U_V$. Hence $\mathcal{M}$ is a regular open star refinement of the regular open covering $\mathcal{U}$.

Remark 1.1. If we change in theorems 1.1. and 1.2 "almost normal" with "almost regular", the corresponding result is well known ([5]). But in general, almost normal nearly paracompact space is not necessarily almost regular. Following is an example.

Example 1.1. Let $X=\{a, b, c\}$ and let τ consist of X, $\varnothing$, $\{a\}$, $\{b\}$ and $\{a, b\}$. Then (X, τ) is almost normal nearly paracompact space. But (X, τ) is not almost regular in as much as $\{b, c\}$ is regular closed set not containing the point a and there exist no disjoint open sets containing $\{a\}$ and $\{b, c\}$ respectively.

(A space X is said to be almost regular iff for any regularly closed set F and any point $x\notin F$, there exist disjoint open sets containing F and x respectively, [5]).

Definition 1.3. An open cover $\mathcal{U}$ is δ-even iff there exists a τ_p^*-open neighbourhood V of the diagonal in $(X\times X, \tau_p)$ (where τ_p is the product topology) such that for each $x\in X$, $V[x]\subset U$, for some $U\in\mathcal{U}$ $(V[x]=\{y:(x, y)\in V\})$ ([1]).

Theorem. 1.3. *Let X be any almost normal nearly paracompact space Then, every regular open cover is δ-even.*

Proof. Let $\{U_\alpha:\alpha\in I\}$ be any regular open covering of X. Let $\mathcal{V}=\{V_\beta:\beta\in J\}$ be any regular open star refinement of $\mathcal{U}$. Let $V=\cup\{V_\beta\times V_\beta:\beta\in J\}$. Then V is τ_p^*-open neighbourhood of the diagonal in $X\times X$. We shall prove that, for each $x\in X$, $V[x]\subset U_\alpha$ for some $\alpha\in I$. If $y\in V[x]$, then $(x, y)\in V$ and hence there exists $\beta\in J$ such that $(x, y)\in V_\beta\times V_\beta$. Then $x\in V_\beta$ and $y\in V_\beta$. Hence $y\in St(x, \mathcal{V})$. Then $V[x]\subset St(x, \mathcal{V})$. Since $\mathcal{V}$ is star refinement of $\mathcal{U}$ therefore there exists $\alpha\in I$ such that $V[x]\subset St(x, \mathcal{V})\subset U_\alpha$. Hence the result.

A space X is said to be nearly Lindelöf iff every regular open covering of X has a countable subcovering ([5]).

Lemma 1.2. *Let X be any almost regular nearly paracompact space. Every regular open cover $\{U_i : i \in I\}$ admits a regular open locally finite refinement $\{V_i : i \in I\}$ such that $\overline{V}_i \in U_i$ for each $i \in I$.*

Proof. Let $\{U_i : i \in I\}$ be any regular open cover of X. By almost regularity, there exists a regular open refinement $\mathcal{A}$ of $\mathcal{U}$ such that for any $A \in \mathcal{A}$, $\overline{A} \subset U_i$ for some $i \in I$. Since X is nearly paracompact there exists a regular open locally finite refinement $\{B_j : j \in J\}$ of $\mathcal{A}$. For each $j \in J$, choose $i(j) \in I$ such that $\overline{B}_j \subset U_{i(j)}$. Let $G_i = \bigcup_{i(j)=i} A_j \subset U_i$. Then $\overline{G}_i = \overline{\bigcup_{i(j)=i} A_j} = \bigcup_{i(j)=i} \overline{A}_j \subset U_i$. Then $\mathcal{V} = \{\alpha(G_i) : i \in I\}$ is a regular open locally finite refinement of $\mathcal{U}$ such that $\overline{\alpha(G_i)} = V_i \subset \overline{U}_i$ for each $i \in I$.

Theorem 1.4. *Let X be any almost regular nearly paracompact space such that there exists a dense nearly Lindelöf (that is nearly Lindelöf set as a subspace of X) set A. Then X is a nearly Lindelöf space.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any regular open cover of X. By preceding lemma, there exists a regular open locally finite refinement $\mathcal{V} = \{V_\alpha : \alpha \in I\}$ of $\mathcal{U}$ such that $\overline{V}_\alpha \subset U_\alpha$ for each $\alpha \in I$. Then $\{A \cap V_\alpha : \alpha \in I\}$ is relatively regular open covering of A, therefore by hypothesis, there exists a countable set $I_0 \subset I$ such that $A = \bigcup \{A \cap V_\alpha : \alpha \in I_0\}$. Then

$$X = \overline{A} = \overline{\bigcup_{\alpha \in I_0} A \cap V_\alpha} = \bigcup_{\alpha \in I_0} \overline{A \cap V_\alpha} \subset \bigcup_{\alpha \in I_0} \overline{V}_\alpha \subset \bigcup_{\alpha \in I_0} U_\alpha, \text{ and thus } X$$

is nearly Lindelöf space.

Theorem 1.5. *Let X be any locally nearly compact and nearly paracompact space. Then $X = \bigcup_{\alpha \in I} V_i$, where V_i are pairwise disjoint and α-nearly Lindelöf sets (that is every regular open in X cover of V_i admits a countable subcovering).*

Proof. Let X be any locally nearly compact and nearly paracompact space. Since X is locally nearly compact, for each $x \in X$ there exists a regular open set U_x containing x, such that $\overline{U}_x$ is α-nearly compact subset of X. Consider the regular open cover $\mathcal{U} = \{U_x : x \in X\}$ of the space X. Since X is nearly paracompact, then there exists a regular open locally finite refinement $\mathcal{V} = \{V_i : i \in I\}$ of $\mathcal{U}$. For $V_0 \in \mathcal{V}$, let $\mathcal{S}_k(V_0)$ be a set of all $V \in \mathcal{V}$, such that there exists a sequence $V_1, V_2, \ldots V_k$ of members of $\mathcal{V}$ and $V_k = V$, $V_i \cap V_{i-1} = \varnothing$.

Let $\mathcal{S}(V_0) = \bigcup_{k=1}^{\infty} \mathcal{S}_k(V_0)$ and $S(V_0) = \bigcup \mathcal{S}(V_0)$. Clearly, the family $\mathcal{S}_k(V_0)$ is finite, hence $K(\mathcal{S}(V_0)) \leqslant \aleph_0$. For V_0, $V_1 \in \mathcal{V}$, $S(V_0)$, $S(V_1)$ are clopen sets, which $S(V_0) \equiv S(V_1)$ or $S(V_0) \cap S(V_1) = \varnothing$. Then $S(V_0) = \bigcup_{V \in \mathcal{S}(V_0)} \overline{V}$, hence $S(V_0)$ is an α-nearly Lindelöf subset, because $S(V_0)$ is a countable union of α-nearly compact subsets. Hence the result.

2. Subsets and nearly paracompact spaces

Definition 2.1. Let X be a topological space, and A subset of X. The set A is α-nearly paracompact iff every regular open cover (in X) of A, has an open (in X) refinement which covers A and is locally finite for every point in X (X-locally finite). The subset A is nearly paracompact iff A is nearly paracompact as a subspace.

Lemma 2.1. *Let A be an α-nearly paracompact subset of a space X. Then A is α-nearly paracompact subset of (X, τ) if and only if A is α-paracompact subset of (X, τ^*).*

Proof. Let A be any α-nearly paracompact subset of (X, τ). Let $\mathcal{C}$ be a basic τ^*-covering of A. Then, $\mathcal{C}$ is also a τ-regular open covering of A hence there exists a regular open X-locally finite refinement $\mathcal{V} = \{V_\beta : \beta \in J\}$ of $\mathcal{C}$, which covers A. Hence A is an α-paracompact subset of (X, τ^*).

Conversely, let $\mathcal{C}$ be τ-regular open covering of A. Then $C = \alpha(C)$ for each $C \in \mathcal{C}$, and $\{\alpha(C) : C \in \mathcal{C}\}$ is τ^*-open covering of A, therefore there exists an X-locally finite τ^*-open refinement $\mathcal{V}$ of $\mathcal{C}$, which covers the set A. Hence A is an α-nearly paracompact subset of the space (X, τ).

Theorem 2.1. *Every τ^*-closed subset of a nearly paracompact space is α-nearly paracompact.*

Proof. Let A be any τ^*-closed subset of a nearly paracompact space (X, τ). Then (X, τ^*) is paracompact ([1]). Thus A is α-paracompact subset of (X, τ^*). Hence by preceding lemma, A is α-nearly paracompact subset of the space (X, τ).

Theorem 2.2. *A regularly closed subset of an α-nearly paracompact is α-nearly paracompact.*

Proof. Let C be any α-nearly paracompact, B regularly closed and $B \subset C$. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any regular open (in X) covering of B. Then $\mathcal{U} \cup \{X \setminus B\}$ is a regularly open cover of C. Since C is α-nearly paracompact there exists an open X-locally finite refinement $\mathcal{V} = \{V_\beta : \beta \in J\}$ of $\mathcal{U} \cup \{X \setminus B\}$ such that $C \subset \{X \setminus B\} \cup \{V_\beta : \beta \in J\}$. It follows that $B \subset \cup \{V_\beta : \beta \in J\}$, and hence B is α-nearly paracompact.

Theorem 2.3. *Let X be an almost regular space, and let A be any α-nearly paracompact subset of X. Then $\overline{A}$ is α-nearly paracompact subset of X.*

Proof. Let A be any α-nearly paracompact subset of X. Let $\mathcal{U}=\{U_\alpha : \alpha\in I\}$ be any regular open cover of $\bar{A}$. For each $x\in A$, there exists U_α containing x.Since X is almost regular, there exists a regularly open (in X) set V_x such that $x\in V_x\subset \overline{V_x}\subset U_\alpha$.

Consider, the regular open covering $\mathcal{V}=\{V_x : x\in A\}$ of A. By hypothesis there exists an X-locally finite family of open sets (in X) $\mathcal{W}=\{W_\beta : \beta\in J\}$ which refines $\mathcal{V}$ and covers A. Then $\{\alpha(W_\beta) : \beta\in J\}$ is an X-locally finite regularly open (in X) family which refines $\{U_\alpha : \alpha\in I\}$ and covers A.

For each $x\in A$, there exists W_β such that $x\in W_\beta$. Since X is almost regular, there exists a regularly open (in X) set V_x^* such that $x\in V_x^*\subset\overline{V_x^*}\subset \alpha(W_\beta)$. Since $\{V_x^* : x\in A\}$ is regularly open covering of α-nearly paracompact subset A, therefore there exists an X-locally finite open (in X) refinement $\{A_\lambda : \lambda\in\Lambda\}$ of $\{V_x^* : x\in A\}$ which covers A. Then, $\bar{A}\subset\overline{\cup\{A_\lambda : \lambda\in\Lambda\}}=\cup\{\bar{A}_\lambda : \lambda\in\Lambda\}$. $\bar{A}_\lambda\subset\overline{V_{x^*}^*}$ for some $x^*\in A$, i.e. $\bar{A}\subset\overline{V_{x^*}^*}\subset\alpha(W_{\beta_0})$ for some $\beta_0\in J$. Thus, $\{\alpha(W_{\beta_0}) : \beta\in J\}$ is an X-locally finite regularly open (in X) refinement of the regular open family $\{U_\alpha : \alpha\in I\}$ which covers $\bar{A}$, hence $\bar{A}$ is α-nearly paracompact subset of the space X.

Theorem 2.4. *In any space the union of a locally finite family of regularly open α-nearly paracompact sets is α-nearly paracompact.*

Proof. Let $\{U_\alpha : \alpha\in I\}$ be any locally finite family of regularly open α-nearly paracompact sets and let $U=\cup\{U_\alpha : \alpha\in I\}$. Let $\{V_\beta : \beta\in J\}$ be any regular open (in X) covering of U. Then, for each α, $\{V_\beta\cap U_\alpha : \beta\in J\}$ is a regular open covering of U_α. Since U_α is α-nearly paracompact, then there exists an X-locally finite family of open sets $\{D_\lambda : \lambda\in K^\alpha\}$ which refines $\{V_\beta\cap U_\alpha : \beta\in J\}$ and covers U_α. Consider the family $\{D_\lambda : \lambda\in K^\alpha : \alpha\in I\}$. Then, this, is an X-locally finite open refinement of $\{V_\beta : \beta\in J\}$ and hence U is α-nealy paracompact.

Corollary 2.1. *In an almost regular space the union of the closures of a locally finite family of regular open α-nearly paracompact sets is α-nearly paracompact.*

Theorem 2.5. *The product of an α-nearly paracompact and an α-nearly compact sets is an α-nearly paracompact.*

Proof. It is similar to the proof of corresponding theorem for nearly paracompact and nearly compact spaces ([5]).

Definition 2.2. A mapping $f : X\to Y$ is said to be almost continuous if the inverse image of every regularly open subset of Y is an open subset of X. f is called almost open (almost closed) if the image of every regularly open (regularly closed) subset of X is an open (a closed) subset of Y, [5].

Theorem 2.6. *Let $f: X \to Y$ be any almost closed, almost continuous and almost open mapping of a space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$. If $A \subset X$ is any α-nearly paracompact subset of X then $f(A) = B \subset Y$ is α-nearly paracompact subset of Y.*

Proof. It is similar to the proof of corresponding theorem for nearly paracompact spaces. ([2]).

REFERENCES

]1] I. Kovačević *A note on nearly paracompact spaces*, To appear in Proceeding of the Belgrade Topological Symposium 1977.

[2] R. Noiri. *Completely continous images of nearly paracompact spaces*, Matematiški vesnik **1** (14) (29), 1977 pp 59—64.

[3] M. K. Singal and S. P. Arya, *On m-paracompcct spaces*, Math. Ann. **181**, 1969, pp 119—133.

[4] M. K. Signal, and A. Mathur, *On nearly compact spases —II*, Bull: U. M. Ital. **4** (9), 1974, pp 679—678.

[5] M. K. Singal and S. P. Arya, *On nearly paracompact spaces*, Matematički vesnik **6** (21) 1969, pp 3—16.

Kovačević Ilija, University of Novi Sad,
Faculty of Technical Science, Department of mathematics,
Veljka Vlahovića 3, 2 1000 Novi Sad, Yugoslavia.