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THEOREMS ON NONLINEAR SUPERPOSITIONS, III: ORDINARY DIFFERENTIAL EQUATIONS

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0. In [1] S. E. Jones and W. F. Ames introduced the concept of nonlinear superposition for ordinary and partial differential equations:

Denote by $S(E)$ the set of all solutions of differential equation (E). If

$$u_1, \ldots, u_m \in S(E) \text{ implies } F(u_1, \ldots, u_m) \in S(E),$$

then the function F is called "connecting function" for (E). It defines a nonlinear superposition for (E).

In a number of papers ([1]—[7]) nonlinear superposition for partial differential equations were investigated.

M. N. Spijker [8] considered a class of ordinary differential equations for which linear superposition is the only kind of superposition which is possible.

In section 1, applying a similar method as in [7], we shall prove the theorem 1.1 which gives sufficient and necessary conditions such that $F(u, v, x)$ is connecting function for equation

$$(E_1) \qquad y'' = f(y', y, x)$$

(f is differentiable function of order 3).

In section 2 we shall formulate an analogous theorem for ordinary differential equations of order $n\,(n \geqslant 2)$.

Some remarks and examples are given in sections 3 and 4.

Remark 0.1. We consider equation (E_1) instead the general equation $\Phi(y'', y', y, x) = 0$ for technical reasons. Similar conclusions can be obtained for this equation.

In further text ψ_i denotes the partial derivative with respect to i-th variable of ψ and $\psi_{ij} = (\psi_i)_j$.

1. Theorem 1.1. *A twice differentiable function* $F(u, v, x)$ *is connecting function for equation* (E_1) *if and only if*:

(i) *equation* (E_1) *has the form*

$$(E_2) \qquad a_1(x)\,(g_y(y, x)\,y'' + g_{yy}(y, x)\,(y')^2 + 2g_{yx}(y, x)\,y' + g_{xx}(y, x)) + 2a_2(x)\,(g_y(y, x)\,y' + g_x(y, x)) + a_3(x)\,g(y, x) = 0$$

(a_1, a_2, a_3 are functions of x, g is a twice differentiable function).

In this case

$$(1.1) \qquad F(u, v, x) = h((C_1 g(u, x) + C_2 g(v, x)), x)$$

where C_1, C_2 are arbitrary constants and the solution with respect to s of the equation

$$(1.2) \qquad g(s, x) = r$$

is given by

$$(1.3) \qquad s = h(r, x);$$

or

(ii) *equation* (E_1) *is of the form*

$$(E_3) \qquad a_1(x)\,(g_y(y, x)\,y'' + g_{yy}(y, x)\,(y')^2 + 2g_{yx}(y, x)\,y' + g_{xx}(y, x)) + 2a_2(x)\,(g_y(y, x)\,y' + g_x(y, x)) + a_3(x)\,g(y, x) + a_4(x) = 0$$

($a_1, \dots, a_4$ are function of x, g is twice differentiable function).

In this case

$$(1.4) \qquad F(u, v, x) = h((C_1 g(u, x) + (1 - C_1)\,g(v, x)), x)$$

(C_1 is arbitrary constant).

Proof. 1°. Sufficient conditions. Let $u, v \in S(E_2)$. Then from (1.1) it follows $g(F, x) = C_1 g(u,x) + C_2 g(v, x)$. Using this, it easy to check that F is also a solution of (E_2).

Similarly we can prove that, if condition (ii) is fulfilled, then F, given by (1.4), is a connecting function for (E_3) and the first part of the theorem is proved.

2° Necessary conditions. Let $u, v \in S(E_1)$ and let $F(u, v, x) \in S(E_1)$. Then we have

$$(1.5) \qquad F_u f(u', u, x) + F_v f(v', v, x) + F_{uu}(u')^2 + 2F_{uv} u' v' + F_{vv}(v')^2 + 2(F_{ux} u' + F_{vx} v') + F_{xx} = f(F_u u' + F_v v' + F_x, F, x).$$

Differentiating (1.5) twice with respect to u' and v' we find $f_{111}\left(\dfrac{dF}{dx}, F, x\right) = 0$ which means that f has the form

$$(1.6) \qquad f(t, s, x) = A(s, x)\,t^2 + B(s, x)\,t + C(s, x)$$

(A, B, C are functions of s, x).

Substituting (1.6) into (1.5) we find:

$$A(F, x) = \frac{F_{uv}}{F_u F_v}, \tag{1.7}$$

$$F_p A(p, x) + F_{pp} = A(F, x)\,(F_p)^2, \tag{1.8}$$

$$F_p B(p, x) + F_{px} = A(F, x) F_p F_x + B(F, x) F_p, \tag{1.9}$$

$$F_u C(u, x) + F_v C(u, x) + F_{xx} = A(F, x)\,(F_x)^2 + 2B(F, x) F_x + C(F, x) \tag{1.10}$$

where $p \in \{u, v\}$.

Furthermore, let

$$A(s, x) = -\frac{G_{ss}(s, x)}{G_s(s, x)} \tag{1.11}$$

where $G(s,x) = \alpha_1(x) \int \exp\left(-\int A(s, x)\,ds\right) ds + \alpha_2(x)$ (α_1, α_2 are functions of x).

From (1.7) it follows $(G(F, x))_{uv} = 0$ and we have $G(F, x) = \varphi(u, x) + \psi(v, x)$, ($\varphi$ and ψ are arbitrary functions in u, x and v, x respectively). Furthermore, from (1.8) we obtain that φ and ψ have the form $\varphi(u, x) = K_u(x)\,(G(u, x) - L(x))$, $\psi(v, x) = K_v(x)\,(G(v, x) - L(x))$, where K_u, K_v, L are arbitrary functions of x.

After substitution

$$g(s, x) = G(s, x) - L(x) \tag{1.12}$$

we find

$$g(F, x) = K_u(x)\, g(u, x) + K_v(x)\, g(v, x). \tag{1.13}$$

Then (1.11) becomes

$$A(s, x) = -\frac{g_{ss}(s, x)}{g_s(s, x)}. \tag{1.14}$$

Differentiating (1.13) with respect to p and x ($p \in \{u, v\}$) and using (1.9) we find $B(F, x) + \dfrac{g_{Fx}(F, x)}{g_F(F, x)} = B(p, x) + \dfrac{g_{px}(p, x)}{g_p(p, x)} + \dfrac{K_p'(x)}{K_p(x)}$.

After substitution $B(s, x) = -\dfrac{H_{sx}(s, x)}{H_s(s, x)}$, we obtain

$$\frac{\partial}{\partial x}\left(\log\left(\frac{g_F(F, x)}{H_F(F, x)}\right)\right) = \frac{\partial}{\partial x}\left(\log\left(\frac{g_p(p, x)}{H_p(p, x)} K_p(x)\right)\right), \tag{1.15}$$

where $p \in \{u, v\}$. From the above it follows

$$\frac{\partial}{\partial x}\left(\log\left(\frac{g_s(s, x)}{H_s(s, x)}\right)\right) = -\frac{a_2(x)}{a_1(x)}, \tag{1.16}$$

where a_1, a_2 are functions of x.

From (1.15) and (1.16) it follows $K_u(x)=C_1$, $K_v(x)=C_2$ (C_1, C_2 are arbitrary constants) and

(1.17) $$g(F, x)=C_1 g(u, x)+C_2 g(v, x).$$

Also we have

(1.18) $$B(s, x)=-\frac{g_{sx}(s, x)}{g_s(s, x)}-\frac{a_2(x)}{a_1(x)}.$$

Using (1.14), (1.17), (1.18) we find that (1.10) becomes

$$F_u\left(C(u, x)+\frac{g_{xx}(u, x)}{g_x(u, x)}+2\frac{a_2}{a_1}\frac{g_x(u, x)}{g_u(u, x)}\right)+F_v\left(C(v, x)+\frac{g_{xx}(v, x)}{g_v(v, x)}+2\frac{a_2}{a_1}\frac{g_x(v, x)}{g_v(v, x)}\right)$$
$$=C(F, x)+\frac{g_{FF}(F, x)}{g_F(F, x)}+2\frac{a_2}{a_1}\frac{g_F(F, x)}{g_F(F, x)}.$$

After substitution

(1.19) $$C(s, x)=\frac{g_{xx}(s, x)}{g_u(s, x)}+2\frac{a_2}{a_1}\frac{g_x(s, x)}{g_s(s, x)}=D(s, x)$$

we get

(1.20) $$F_u D(u, x)+F_v D(v, x)=D(F, x).$$

Differentiating (1.20) with respect to u and v and using (1.7), (1.8), (1.14) we find

(1.21) $$D_F^{-}(F, x)+D(F, x)\frac{g_{FF}(F, x)}{g_F(F, x)}=D_p(p, x)+D(p, x)\frac{g_{pp}(p, x)}{g_p(p, x)}=-\frac{a_3(x)}{a_1(x)}$$

where $p\in\{u, v\}$ and a_3 is a function of x.

Then we have

(1.22) $$D(s, x)=-\frac{a_3(x)\, g(s, x)+a_4(x)}{a_1(x)\, g_s(s, x)}$$

where a_4 is a function of x.

From (1.19) and (1.22) we find

(1.23) $$C(s, x)=-\frac{a_1(x)\, g_{xx}(s, x)+2a_2(x)\, g_x(s, x)+a_3(x)\, g(s, x)+a_4(x)}{a_1(x)\, g_s(s, x)}.$$

Furthermore, from (1.17), (1.20), (1.22) it follows that the following condition must be satisfied

(1.24) $$(C_1+C_2-1)\, a_4(x)=0.$$

The following two cases are possible:

1° $C_1+C_2\neq 1$. Then $a_4(x)=0$ and from (1.17) it follows that F is of the form (1.1). Also, from (1.6), (1.14), (1.18), (1.23) we conclude that equation (E_1) has the form (E_2).

2° $C_1+C_2=1$. Then from (1.17) we find that F is of the form (1.4). Also, from (1.6), (1.14), (1.18), (1,23) it follows that equation (E_1) has the form (E_3).

This proves the theorem.

2. In this section we shall formulate a theorem analogous to Theorem 1.1 for ordinary differential equations of order n.

Theorem 2.1. *A differentiable function of order,* $F(u_1, \ldots, u_m, x)$ $(m \leqslant n)$ *is a connecting function for the equation*

$$(E_4) \qquad y^{(n)} = \Phi(y^{(n-1)}, \ldots, y', y, x)$$

$(n \geqslant 2)$ *if and only if one of the following two conditions is satisfied:*

(i) *equation* (E_4) *is of the form*

$$(E_5) \qquad a_0(x)\frac{d^n g(y, x)}{dx^n} + \cdots + a_{n-1}(x)\frac{dg(y, x)}{dx} + a_n(x)g(y, x) = 0,$$

$(a_0, \ldots, a_n$ *are functions of* x, g *is diferentiable function of order* n). *Then* F *has the form*

$$(2.1) \qquad F(u_1, \ldots, u_m, x) = h((C_1 g(u_1, x) + \cdots + C_m g(u_m, x)), x)$$

where $C_1, \ldots, C_m$ *are arbitrary constants and the solution with respect to* s *of the equation* $g(s, x) = r$ *is given by* $s = h(r, x)$.

(ii) *equation* (E_4) *has the form*

$$(E_6) \qquad a_0(x)\frac{d^n g(y, x)}{dx^n} + \cdots + a_{n-1}(x)\frac{dg(y, x)}{dx} + a_n(x)g(y, x) + a_{n+1}(x) = 0,$$

where $a_0, \ldots, a_{n+1}$ *are functions of* x, g *is differentiable function of order* n. *In this case.*

$$(2.2) \qquad F(u_1, \ldots, u_m, x) = h((C_1 g(u_1, x) + \cdots + C_{m-1} g(u_{m-1}, x) + g(u_m, x)(1 - C_1 - \cdots - C_{m-1})), x)$$

$(C_1, \ldots, C_{m-1}$ *are arbitrary constants*).

3. Remarks 1°. It easy to see that substitution $Y = g(y, x)$ reduces equations (E_5) and (E_6) to linear homogeneous and nonhomogeneous ordinary differential equation of order n, respectively. This means that the equations with connecting functions of the form $F(u_1, \ldots, u_m, x)$ can be transformed to linear differential equations.

2° From the theorem 2.1 it follows:

General solution of the equation (E_5) is of the form:

$$g(y, x) = C_1 g(u_1, x) + \cdots + C_n g(u_n, x)$$

where $u_1, \ldots, u_n$ are particular solutions of this equation.

Also, the general solution of equation (E_6) is of the form

$$g(y, x) = C_1 g(u_1, x) + \cdots + C_n g(u_n, x) + g(u_{n+1}, x)(1 - C_1 - \cdots - C_n)$$

where $u_1, \ldots, u_{n+1}$ are particular solutions of this equation and $C_1, \ldots, C_n$ are arbitrary constants.

3° If we suppose that function F does not depend on x we obtain the following result:

Function $F(u_1, \ldots, u_m)$ is a connecting function for equation (E_5) if and only if (E_5) is of the form

$$\text{(i)} \qquad a_0(x)\frac{d^n g(y)}{dx^n} + \cdots + a_{n-1}(x)\frac{dg(y)}{dx} + a_n(x) g(y) = 0.$$

In this case $F(u_1, \ldots, u_m) = g^{-1}(C_1 g(u_1) + \cdots + C_m g(u_m))$ ($C_1, \ldots, C_m$ are arbitrary constants, g^{-1} is the inverse of g); or

$$\text{(ii)} \qquad a_0(x)\frac{d^n g(y)}{dx^n} + \cdots + a_{n-1}(x)\frac{dg(y)}{dx} + a_n(x) g(y) + a_{n+1}(x) = 0.$$

In this case connecting function has the form $F(u_1, \ldots, u_m) = g^{-1}(C_1 g(u_1) + \cdots + C_{m-1} g(u_{m-1}) + g(u_m)(1 - C_1 - \cdots - C_{m-1})$. ($C_1, \ldots, C_{m-1}$ are arbitrary constants).

4° Theorem 1.1 can be formulated in the following way:

A twice differentiable function $F(u, v, x)$ is a connecting function for the equation (E_1) if and only if one of the following two conditions is fulfilled:

(i) equation (E_1) has the form

$$(E_8) \qquad y'' + A(y, x)(y')^2 + 2(B(y, x) + a(x))y' + C(y, x) + 2a(x) D(y, x) + b(x) E(y, x) = 0$$

(a, b are functions of x). A, B, C, D, E are functions of y, x which satisfy the following system of partial differential equations:

$$(3.1) \qquad E_y = 1 - AE, \quad E_x + ED_y = E_y D, \quad D_y = B - DA, \quad D_x = C - DB.$$

In this case all connecting functions F can be obtained from

$$\exp\left(\int \frac{dF}{E(F, x)}\right) = C_1 \exp\left(\int \frac{du}{E(u, x)}\right) + C_2 \exp\left(\int \frac{dv}{E(v, x)}\right)$$

(C_1, C_2 are arbitrary constants);

(ii) equation (E_1) is of the form

$$(E_9) \qquad y'' + A(y, x)(y')^2 + 2(B(y, x) + a(x))y' + C(y, x) + 2a(x) D(y, x) + b(x) E(y, x) + c(x) G(y, x) = 0$$

(a, b, c are functions of x). A, B, C, D, E, G are functions of y, x which satisfy the following system of partial differential equations:

(3.2) $E_y = 1 - AE, \quad E_x + ED_y = E_y D, \quad D_y = B - DA, \quad D_x = C - DB, \quad G(EG)_y = 1.$

In this case all connecting functions F can be found from

$$\exp\left(\int \frac{dF}{E(F, x)}\right) = C_1 \exp\left(\int \frac{du}{E(u, x)}\right) + (1 - C_1)\exp\left(\int \frac{dv}{E(v, x)}\right)$$

(C_1 is arbitrary constant).

4. Examples 1°. Equation

$$y'' + \frac{A''(yx)}{A'(yx)} x (y')^2 + 2\left(\frac{A''(yx)}{A'(yx)} y + \frac{1}{x} + a(x)\right) y' + \frac{A''(yx)}{A'(yx)} \frac{y^2}{x} + 2a(x)\frac{y}{x}$$
$$+ b(x)\frac{A(yx)}{A'(yx)}\frac{1}{x} = 0$$

has the connecting function $F(u, v, x) = \frac{1}{x} A^{-1}(C_1(A(ux) + C_2 A(vx))$.

Also the general solution of this equation is of the form $y = \frac{1}{x} A^{-1}(C_1 A(ux) + + C_2 A(vx))$, where C_1, C_2 are arbitrary constants and u and v are particular solutions of the above equation.

2° Equation

$$y'' + \frac{2A''(t)}{A'(t)} y (y')^2 + 2\frac{(y')^2}{y} + 2\left(\frac{2xA''(t)}{A'(t)} + a(x)\right) y' + \frac{2x^2 A''(t)}{yA'(t)}$$
$$+ b(x)\frac{A(t)}{A'(t)} = 0$$

where $t = x^2 + y^2$, has connecting function given by

$$F(u, v, x) = (A^{-1}(C_1 A(x^2 + u^2) + C_2 A(x^2 + v^2)) - x^2)^{1/2}.$$

3° For the equation

(4.1) $$\sum_{k=0}^{n} a_{n-k}(x)\left(\sum_{j=0}^{k} \binom{k}{j} A^{(k-j)}(x)\, y^{(j)} + B^{(k)}(x)\right) = 0$$

all connecting functions are of the form

(4.2) $$F(u_1, \ldots, u_m, x) = \sum_{k=1}^{m} C_k u_k + \frac{B(x)}{A(x)}\left(\sum_{k=1}^{m} C_k - 1\right).$$

(A, B are given functions, $C_1, \ldots, C_m$ are arbitrary constants).

4° Equation

$$\sum_{k=0}^{n} a_{n-k}(x)\left(\sum_{j=0}^{k}\binom{k}{j}A^{(k-j)}(x)\,y^{(j)}+B^{(k)}(x)\right)+a_{n+1}(x)=0 \tag{4.3}$$

has connecting function

$$F(u_1, \ldots, u_m, x)=\sum_{k=1}^{m-1} C_k(u_k-u_m)+u_m \tag{4.4}$$

(A, B are given function, $C_1, \ldots, C_{m-1}$ are arbitrary constants).

5° M. N. Spijker in [8] considered a class of ordinary differential equations of the form (E_4) such that function Φ satisfies the following condition:

$$\lim_{s\to+\infty} M(s, y, x)\,s^{-n}=0, \tag{4.5}$$

where $M(s, y, x)=\max\{\,|\Phi(t_{n-1}, \ldots, t_1, y, x)|\;|\;|t_i|<s^i,\ (i=1, \ldots, n-1)\}$.

From the theorem 1.1. it follows that the equations for which $F(u, v, x)$ is a connecting function, are of the form (E_2) or (E_3). If we suppose that the condition (4.5) is fulfilled, we obtain $\left|\dfrac{g_{yy}(y, x)}{g_y(y, x)}\right|=0$, which gives $g(y, x)=$ $=A(x)y+B(x)$. In this case equations (E_2) and (E_3) reduce to equations (4.1) and (4.3) for $n=2$, respectively. Then connecting functions are given by (4.2) and (4.4) for $n=2$, $m=2$. This is, in fact, a more general result from Spijker's for $n=m=2$.

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