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ON BOREL, BAIRE AND LEBESGUE SETS

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Abstract. **In this paper it is shown that there are subsets of the set R of all real numbers which are neither Baire nor Lebesgue sets. From this it is easily derived that there are subsets of R which are Baire sets and not Lebesgue sets and that there are subsets of R which are Lebesgue sets and not Baire sets.**

In what follows R stands for the set of all real numbers. The smallest (w.r.t. $\subseteq$) σ-algebra B containing as elements every open interval of R is called the *σ-algebra of Borel sets*. Accordingly, a subset D of R is called a *Borel set* if and only if $D\in B$.

It can be readily verified that there are $\overline{\overline{R}}=2^{\aleph_0}$ Borel sets.

The smallest (w.r.t. $\subseteq$) σ-algebra A containing as elements every open interval of R and every set of first category is called the *σ-algebra of Baire sets*. Accordingly, a subset E of R is called a *Baire set* if and only if $E\in A$. In general, a subset of R is a Baire set if and only if it is a symmetric difference of an open subset of R and a First Category set.

The smallest (w.r.t. $\subseteq$) σ-algebra L containing as elements every open interval of R and every set of measure zero is called the *σ-algebra of Lebesgue sets*. Accordingly, a subset H of R is called a *Lebesgue set* if and only if $H\in L$.

Theorem 1. *Every subset of R is a disjoint union of a Baire set (in act, a F irst Category set) and a Lebesgue set (in fact, a Measure Zero set).*

Proof. For $n=0, 1, 2, \ldots$, let V_n denote an open set whish covers the set of all rational numbers such that the length of V_n is equal to 2^{-n} (by the length of V_n we mean the total sum of the lenghts of the open intervals which are the components of V_n). Clearly, the complement V_n' of V_n is a nowhere dense set. Obviously, $\bigcup_{i\in\omega} V_n'$ is a First Category set and $\bigcap_{i\in\omega} V_n$ is a Measure Zero set. Since $R=(\bigcup_{i\in\omega} V_n')\cup(\bigcap_{i\in\omega} V_n)$, we see that R satisfies the conclusion of the Theorem. But then since every subset of a First Category set is of first category and every subset of a Measure Zero set is of measure zero, we see that every subset of R satisfies the conclusion of the Theorem.

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In what follows, B, A, L stand respectively for the sets of all Borel, Baire and Lebesgue sets of R.

Theorem 2. *Every Borel set is both a Baire and a Lebesgue set. However, there exist sets which are both Baire and Lebesgue but not Borel.* Thus,

$$B \subseteq (A \cap L) \quad \text{and} \quad (A \cap L) - B \neq \varnothing$$

Proof. Since both A and L are σ-algebras containing as elements every open interval of R, and since B is the smallest (w.r.t, $\subseteq$) such σ-Algebra, we see that $B \subseteq (A \cap L)$.

On the other hand, every subset of a Cantor ternary set C is a Baire set of first category as well as a Lebesgue set of measure zero. Since $\overline{\overline{C}} = \overline{\overline{R}}$ we see that there are $2^{\overline{\overline{R}}}$ subsets of R which are both Baire and Lebesgue. However, as mentioned earlier there are altogether $\overline{\overline{R}}$ Borel sets. Thus, indeed

$$(A \cap L) - B \neq \varnothing .$$

Theorem 3. *There exist subsets of R which are neither Baire nor Lebesgue sets.*

Proof. Let us observe [2, pp. 514] that there are subsets X of R such that X as well as the complement of X has a nonempty intersection with every nondenumerable closed subset of R. Let K be such a subset of R. Since K as well as its complement K' has a nonempty intersection with every nondenumerable closed subset of R, we see that the Lebesgue outer measure of $K \cap [0, 1]$ as well as $[0, 1] - (K \cap [0, 1])$ is equal to 1. Consequently, K is not a Lebesgue set.

Next we show that K is not a Baire set. First, let us observe that K cannot be a First Category set. Indeed, as mentioned in [1], a First Category set misses a Cantor type set (which is a nondenumerable closed subset of R) in every nonempty open interval of R, whereas K must have a nonemptv intersection with every nondenumerable closed subset of R. Now, assume on the contrary that K is a Baire set. Thus, $K = G \oplus F$ where G is an open subset of R and F is a First Category set. In view of our observation above, G is nonempty and consequently $G - F$ (and a fortiori $G \oplus F$) contains a Cantor type set. But this contradicts the fact that K' must have a nonempty intersection with $G \oplus F$. Thus, our assumption is false and K is not a Baire set.

Theorem 4. *There exist Baire sets which are not Lebesgue sets and there exist Lebesgue sets which are not Baire sets.*

Proof. In view of Theorem 3, let K be a subset of R such that K is neither a Baire nor a Lebesgue set. From Theorem 1 it follows that $K = K_0 \cup K_1$ where K_0 is a set of measure zero (and therefore a Lebesgue set) and K_1 is a First Category set (and therefore a Baire set). Clearly, K_1 cannot be a Lebesgue set since otherwise K would be a Lebesgue set. Similarly, K_0 cannot be a Baire set since otherwise K would be a Baire set. Thus, K_0 is a Lebesgue set which is not a Baire set.

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