A CONVOLUTION THEOREM WITH APPLICATIONS TO SOME DIVISOR FUNCTIONS

Aleksandar Ivić

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1. Introduction

The convolution of two arithmetical functions \( f(n) \) and \( g(n) \) (or Dirichlet convolution, to distinguish it from unitary and other possible arithmetical convolutions) is the function

\[
(1.1) \quad h(n) = \sum_{d|n} f(n/d) g(d) = \sum_{d|n} g(n/d) f(d),
\]

where the sum is taken over all positive divisors of \( n \). A common procedure in dealing with the asymptotic formula for the sum \( \sum_{n \leq x} h(n) \) is to express \( h(n) \) as a convolution of \( f(n) \) and \( g(n) \) and to derive the asymptotic formula for \( \sum_{n \leq x} h(n) \) from the asymptotic formulas for \( \sum_{n \leq x} f(n) \) and \( \sum_{n \leq x} g(n) \). Such convolution methods were investigated by many authors, and notably by J. P. Tull who in [13] and [14] proved two theorems for the even more general case of the Stieltjes convolution \( \int A(x/u) dB(u) \).

This paper contains two convolution theorems with sharp error terms, of which Theorem 1 is very general, while Theorem 2 may be regarded as a special case of Theorem 1 when \( g(n) = \mu(n) \). Theorem 2 gives also the error term under the assumption that the famous Riemann hypothesis about the non-trivial zeros of the zeta function is true.

In the formulation of both theorems instead of the convolution (1.1) we use

\[
(1.2) \quad h(n) = \sum_{d^k|n} f(n/d^k) g(d).
\]
which may be reduced at once to the form (1.1) by setting \((k = \text{a fixed integer})\)

\[
G(n) = \begin{cases} 
    g(m) & \text{if } n = m^k \\
    0 & \text{else if } n \neq m^k.
\end{cases}
\]

The reason for introducing (1.2) lies in the nature of applications of Theorem 2, since many divisor functions \(h(n)\) may be expressed as \(h(n) = \sum_{d \mid n} \mu(d) f(n/d^k)\), so that Theorem 2 is readily applicable. A number of these
applications is given in Section 3.

For the more general Theorem 1 some properties of slowly oscillating functions are needed. By a slowly oscillating (also called slowly varying) function we shall mean a positive function \(L(x)\) defined for \(x > 0\) and continuous for \(x \geq x_0 > 0\), such that for every \(c > 0\)

\[
\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1.
\]

J. Karamata in [4] characterized such functions in the form

\[
L(x) = a(x) \exp \left( \int_{x_0}^{x} \delta(t) \frac{1}{t} \, dt \right)
\]

where \(a(x)\) and \(\delta(x)\) are continuous for \(x \geq x_0\), \(a(x) \to a_0 > 0\) and \(\delta(x) \to 0\) as \(x \to \infty\). Slowly oscillating functions naturally arise in number theory since most of the functions like \(\log^A x\), \(\log \log x\), \(\exp(C \log^B x)\) (for \(B < 1\)) that appear in the asymptotic formulas for arithmetic functions are slowly oscillating. For a comprehensive account of slowly oscillating and the more general slowly varying functions see [8].

2. Statement and proof of theorems

**Theorem 1.** Let \(f(n)\) be an arithmetical function for which

\[
\sum_{n \leq x} f(n) = \sum_{i=1}^{l} c_i x^{a_i} L_i(x) + O(x^a), \quad \sum_{n \leq x} |f(n)| = O(x^{a_1} P(x)),
\]

where \(a_1 \geq a_2 \geq \ldots \geq a_l \geq 1/k > a \geq 0\), \(c_1, \ldots, c_l\) are constants, \(k \) is a fixed natural number, \(L_1(x), \ldots, L_l(x)\) are slowly oscillating functions, and \(P(x)\) is a non-decreasing slowly oscillating function. Let further \(g(n)\) be an arithmetical function for which

\[
\sum_{n \leq x} g(n) = O(x^b N(x)) \quad \text{for some } 0 \leq b \leq 1, \quad \sum_{n \leq x} |g(n)| = O(x),
\]

\(N(x)\) is a slowly oscillating function of the form \(N(x) = \exp(C \omega(x))\), \(\omega(x) = \int_{x_0}^{x} \tau(t) t^{-1} \, dt\), \(\tau(x)\) is continuous and positive for \(x \geq x_0\), \(\lim_{x \to \infty} \tau(x) = 0\),
\[
\lim_{x \to \infty} P(x) \exp(A \omega(x)) = 0 \quad \text{for every} \quad A < 0, \quad \text{and if} \quad b = 1, \quad C \quad \text{is negative, and if} \quad 0 < b < 1, \quad C \quad \text{is positive.}
\]

If \( h(n) = \sum_{d \mid n} f(n/d^k)g(d) \) then there exist functions \( Q_1(x), \ldots, Q_l(x) \) such that \( Q_i(x) = O(x^\varepsilon) \) for every \( \varepsilon > 0 \) and \( i = 1, \ldots, l \) and

\[
\sum_{n \leq x} h(n) = \sum_{i=1}^l c_i x^{\alpha_i} Q_i(x) + \Delta(x),
\]

where in the case \( b = 1 \) \( \Delta(x) = O(x^{1/k} \exp(D \omega_1(x))) \), \( \omega_1(x) = \int_{x_1}^x \eta(t^{1/k}) t^{-1} dt \) with \( x_1 = x_0^{1/u} \), \( D < 0 \) for every \( u < 1/k \). In the case \( 0 < b < 1 \) we have \( \Delta(x) = O(x^c \exp(D \omega(x))) \), where \( D > 0 \) and \( c = (a_1 - ab)/(a_1 a - ak + 1 - b) \).

**Proof.** Let \( y, z > 1 \) and \( yz = x \). Then using (1.3) we get

\[
\sum_{n \leq x} h(n) = \sum_{m \leq y} \sum_{n \leq x/m} G(m)f(n/m) = \sum_{m \leq y} G(m) \sum_{n \leq x/m} f(n) = S_1 + S_2 - S_3.
\]

so that we obtain

\[
S_1 = \sum_{m \leq y} G(m) \sum_{n \leq x/m} f(n) = \sum_{i=1}^l c_i x^{\alpha_i} \sum_{m \leq y} G(m)m^{-\alpha_i}L_i(x/m) + O\left(x^{a} \sum_{m \leq y} |G(m)m^{-a}\right) = \sum_{i=1}^l c_i x^{\alpha_i} Q_i(x) + O(x^a y^{1/k-a}),
\]

where \( y = y(x) \) will be suitably chosen later, and where we have set

\[
Q_i(x) = \sum_{m \leq y} G(m)m^{-\alpha_i}L_i(x/m) = \sum_{m \leq y}^\infty G(m)m^{-k\alpha_i}L_i(x/m^k).
\]

**i) The case \( b = 1 \).** If \( b = 1 \) then \( N(x) \) it decreasing and therefore for \( n \leq z \) we have \( N((x/n)^{1/k}) \leq N(y^{1/k}) \) which gives

\[
S_2 = \sum_{n \leq z} f(n) \sum_{m \leq x/n} G(m) = O(x^{1/k} N(y^{1/k}) \sum_{n \leq z} |f(n)| n^{-1/k}) = \]

\[
O(x^{1/k} z^{a_1 - 1/k} P(z) N(y^{1/k})) = O(x^{a_1} y^{1/k-a_1} P(x/y) N(y^{1/k})).
\]

\[
S_3 = \sum_{m \leq y} G(m) \sum_{n \leq z} f(n) = O(z^{a_1} P(z) y^{1/k} N(y^{1/k})) = O(x^{a_1} y^{1/k-a_1} P(x/y) N(y^{1/k})).
\]
Therefore we obtain

\[(2.5) \quad \sum_{n \leq x} h(n) = \sum_{i=1}^{l} c_{i} x^{q_{i}} Q_{i}(x) + O(x^{a} y^{1/k-a}) + O(x^{a_{1}} y^{1/k-a_{1}} P(x/y) N(y^{1/k})).\]

Let now \(0 < u < 1/k\) and choose \(y = x(N(x^{u}))^{1/(a_{1} - a)}\), so that \(y < x\) for \(x > x_{p}\). From (1.5) it follows that \(L(x) = O(x^{u})\) for every \(\varepsilon > 0\) if \(L(x)\) is slowly oscillating, which gives \(x^{u} \leq x^{1/k - \varepsilon} \leq y^{1/k}\) for \(0 < \varepsilon < 1/k - u\), so that \(N(y^{1/k}) \leq N(x^{u})\). This means that the error terms in (2.5) may be written as

\[O \left( x^{1/k} (N(x^{u}))^{1/(a_{1} - a)} \left( 1 + \frac{N(y^{1/k})}{N(x^{u})} P(x/y) \right) \right) = O \left( x^{1/k} (N(x^{u}))^{1/(a_{1} - a)} P(x^{u}) \right),\]

since \(x/y < x^{u}\) for \(x\) large enough. If \(C_{1} = (C/k - Ca)/(a_{1} - a)\), then for every \(A < 0\)

\[(N(x^{u}))^{1/(a_{1} - a)} P(x^{u}) = P(x^{u}) \exp(A \omega(x^{u})) \exp((C_{1} - A) \omega(x^{u})) = O(\exp(D \omega(x^{u}))),\]

where \(D = (C_{1} - A)u\), \(\omega_{1}(x) = \int_{x_{1}}^{x} \eta(t^{u}) t^{-1} dt\), \(x_{1} = x_{0}^{1/u}\), since

\[\lim_{x \to \infty} P(x^{u}) \exp(A \omega(x^{u})) = 0.\]

\[\text{ii) The case } 0 \leq b < 1.\text{ If } 0 \leq b < 1 \text{ then } N(x) \text{ is increasing and therefore } N(x^{1/k} n^{-1/k}) \leq N(x), \text{ so that}\]

\[S_{2} = O(x^{b/k} N(x) \sum_{n \leq z} |f(n)| n^{-b/k}) = O(x^{b/k} N(x) P(x) z^{a_{1} - b/k}) = O(x^{a_{1} y^{b/k - b_{1}}} N(x) P(x)),\]

and the same estimate holds for \(S_{3}\), which yields

\[(2.6) \quad \sum_{n \leq x} h(n) = \sum_{i=1}^{l} c_{i} x^{q_{i}} Q_{i}(x) + O(x^{a} y^{1/k-a}) + O(x^{a_{1}} y^{b/k-a_{1}} N(x) P(x)).\]

If \(D > C\) then

\[N(x) P(x) = \exp(D \omega(x)) \exp((C - D) \omega(x)) P(x) = O(\exp(D \omega(x))),\]

since \(\lim_{x \to \infty} P(x) \exp(A \omega(x)) = 0\) for \(A = C - D < 0\). Taking now \(y = x^{u}\) where

\[q = k(a_{1} - a)/(1 - b + k(a_{1} - a))\]

we obtain finally

\[\sum_{n \leq x} h(n) = \sum_{i=1}^{l} c_{i} x^{q_{i}} Q_{i}(x) + O(x^{c} \exp(D \omega(x))),\]

where \(c = (a_{1} - ab)/(1 - b + k(a_{1} - a))\), as stated in the theorem,
Concerning the functions \( Q_t(x) \) it follows from (2.4)

\[
Q_t(x) = O \left( \sum_{m \leq x} G(m) |m^{-a_t} L_t(x/m)| \right) = O \left( x^t \sum_{m \leq x} |G(m)| m^{-a_t} \right) = O(x^t),
\]

since \( L_t(x) = O(x^t) \), and the second sum above is bounded. It may be further shown that

\[
\lim_{x \to \infty} \frac{Q_t(x)}{L_t(x)} = \sum_{n=1}^{\infty} \frac{g(n)}{n} \frac{1}{n^{-k}a_t},
\]

which means that \( Q_t(x) \) is slowly oscillating if it is continuous and the above limit is positive, since it is then asymptotic to a slowly oscillating function. A more detailed discussion is omitted, since in many applications to divisor problems the functions \( Q_t(x) \) turn out to be polynomials in \( \log x \).

**Theorem 2.** Let \( f(n) \) be an arithmetical function for which

\[
\sum_{n \leq x} f(n) = \sum_{i=1}^{t} x^{a_i} P_i(\log x) + O(x^{a_1} \log^{r} x),
\]

where \( a_1 \geq a_2 \geq \cdots \geq a_t > 1/k > a > 0 \), \( r \geq 0 \), \( P_1(t) \), \( \ldots \), \( P_t(t) \) are polynomials in \( t \) with degrees not exceeding \( t \), and \( k \) is a fixed natural number.

If \( h(n) = \sum_{d^k \mid n} \mu(d) f(n/d^k) \) where \( \mu(n) \) is the Möbius function, then

\[
\sum_{n \leq x} h(n) = \sum_{i=1}^{t} x^{a_i} R_i(\log x) + \Delta(x),
\]

where \( R_1(t), \ldots, R_t(t) \) are polynomials in \( t \), and for some \( D > 0 \)

\[
\Delta(x) = O(x^{1/k} \exp(-D \log^{3/5} x \cdot (\log \log x)^{-1/5})).
\]

If the Riemann hypothesis is true, then for some \( D > 0 \)

\[
\Delta(x) = O(x^c \exp(D \log x \cdot (\log \log x)^{-1})), \quad c = (2a_1 - a)/(2ka_1 - 2ka + 1).
\]

**Proof.** Theorem 2 is a special case of Theorem 1 when \( g(n) = \mu(n) \), \( c_i L_i(x) = P_i(\log x) \), \( P(x) = \log^{r} x \). For \( \sum_{n \leq x} \mu(n) \) we use the following best-known estimate due to A. Walfisz [15]:

\[
M(x) = \sum_{n \leq x} \mu(n) = O(x \exp(-C \varepsilon(x))),
\]

where \( C > 0 \) and from now on \( \varepsilon(x) \) denotes \( \varepsilon(x) = \log^{3/5} x \cdot (\log \log x)^{-1/5} \). This corresponds to the case \( b = 1 \) of Th. 1; if the Riemann hypothesis that all nontrivial zeros of \( \zeta(s) \) lie on \( s = 1/2 + it \) is true, then as shown in [12], Ch. XIV

\[
M(x) = \sum_{n \leq x} \mu(n) = O(x^{1/2} \exp(C \omega(x))),
\]
where \( C > 0 \) and from now on \( \omega_1(x) \) denotes \( \omega(x) = \log x \cdot (\log \log x)^{-1} \), and this corresponds to the case \( b = \frac{1}{2} \) of Th. 1. If one could prove for some \( 1/2 < b < 1 \) \( M(x) = O(x^b) \), then Th. 1 would give for some \( s \geq 0 \) \( \Delta(x) = O(x^c \log^s x) \), where \( c = (a_i - ab)/(a_i - ak + 1 - b) \). It should be noted that

\[
c_i \sum_{m \leq y} G(m) m^{-\alpha_i} L_i(x/m) = c_i \sum_{m = 1}^{\infty} \mu(m) m^{-\alpha_i} L_i(x/m^k) -
- c_i \sum_{m^k > y} \mu(m) m^{-\alpha_i} L_i(x/m^k),
\]

and that \( L_i(x/m^k) \) can be written as a polynomial in \( \log x \), so that

\[
c_i \sum_{m = 1}^{\infty} \mu(m) m^{-\alpha_i} L_i(x/m^k) = R_i(\log x),
\]

where \( R_i(t) \) is a polynomial in \( t \), and it remains to show that sums of the type \( \sum_{m > y^{1/k}} \mu(m) m^{-\alpha_i} \log^4 m \) contribute to the error term. If we set \( y^{1/k} = \nu \), \( \alpha_i = c > 1 \), then

\[
\sum_{m > \nu} \mu(m) m^{-\epsilon} \log^4 m = \int_{\nu}^{\infty} t^{-\epsilon} \log^4 t \cdot dM(t) =
= \nu^{-\epsilon} M(\nu) \log^4 \nu + O\left( \int_{\nu}^{\infty} |M(t)| t^{-\epsilon - 1} \log^4 t \cdot dt \right).
\]

If we use (2.12) then \( \exp(-C \epsilon(x)) \) is decreasing for \( x \geq x_1 \) and thus

\[
\sum_{m > \nu} \mu(m) m^{-\epsilon} \log^4 m = O(\nu^{1-\epsilon} \exp(-C \epsilon(\nu)) \log^4 \nu) +
+ O\left( \exp(-C \epsilon(\nu)) \int_{\nu}^{\infty} t^{-\epsilon} \log^4 t \cdot dt \right) = O(\nu^{1-\epsilon} \exp(C \epsilon(\nu)) \log^4 \nu).
\]

If we use (2.13) then \( x^{-1/2} \exp(C \omega(x)) \) is decreasing for \( x \geq x_2 \) and so

\[
\sum_{m > \nu} \mu(m) m^{-\epsilon} \log^4 m = O(\nu^{1/2-\epsilon} \exp(C \omega(\nu)) \log^4 \nu) +
+ O(\nu^{-1/2} \exp(C \omega(\nu)) \int_{\nu}^{\infty} t^{-\epsilon} \log^4 t \cdot dt) = O(\nu^{1/2-\epsilon} \exp(C \omega(\nu)) \log^4 \nu).
\]

The remaining details of the proof are the same as in Theorem 1; note that

\[
\omega_1(x) \sim u^{-2/5} \epsilon(x).
\]
3. Applications

1. Let first $F_k(s) = \sum_{n=1}^{\infty} f_k(n) n^{-s} = \zeta^k(s)/\zeta(2s)$ for $k \geq 2$. Then we have

$$f_k(n) = \sum_{d^2 \mid n} \mu(d) \tau_k(n/d^2)$$

where $\tau_k(n)$ is the number of representations of $n$ as a product of $k$ factors and

$$\sum_{n=1}^{\infty} \tau_k(n) n^{-s} = \zeta^k(s).$$

Since $\tau_k(n)$ is multiplicative and

$$\tau_k(p^a) = \binom{a+k-1}{k-1} \binom{a+k-3}{k-1} \frac{2a+k-2}{a} \binom{a+k-3}{k-2},$$

then $f_k(n)$ is also multiplicative and

$$f_k(p^a) = \binom{a+k-1}{k-1} \binom{a+k-3}{k-1} \frac{2a+k-2}{a} \binom{a+k-3}{k-2}.$$

For special values of $k$ this gives the following well-known arithmetical functions:

a) for $k = 2$ we have $f_2(p^a) = 2$ so that $f_2(n) = \sum_{d^2 \mid n, (d,n/d) = 1} 1$,

b) for $k = 3$ we have $f_3(p^a) = 2a+1$ so that $f_3(n) = \tau(n^2)$, where $\tau(n)$ is the number of divisors of $n$,

c) for $k = 4$ we have $f_4(p^a) = (a+1)^2$ so that $f_4(n) = \tau^2(n)$.

It is known (see [12], Ch. XII) that

$$\sum_{n \leq x} \tau_k(n) = xP_{k-1}(\log x) + O(x^{\frac{2}{k}}).$$

where $P_{k-1}(t)$ is a polynomial of degree $k-1$ in $t$, $x_k > (k-1)/2k$, for $k \geq 4$ and every $\varepsilon > 0$ $x_k \leq (k-1)/(k+2)+\varepsilon$, $x_3 \leq 346/1067$, $x_4 \leq 5/11$ (see [5] and [4]). It has been conjectured (see [12], Ch. XII) that for every $\varepsilon > 0$ and $k \geq 2$ one has $x_k = (k-1)/2k+\varepsilon$. If we suppose that $x_k < 1/2$ (so far this has been shown to be true only when $k = 2$ and $k = 3$), then Th. 2 may be applied at once to give

$$\sum_{n \leq x} f_k(n) = xH_{k-1}(\log x) + \Delta_k(x), \quad \Delta_k(x) = O(x^{1/2} \exp(-C_k\varepsilon(x))),$$

where $H_{k-1}(t)$ is a polynomial of degree $k-1$ in $t$ whose coefficients may be found for example by residues, and $C_k > 0$ is a constant depending on $k$.

If besides $x_k < 1/2$ we assume the truth of the Riemann hypothesis, then the second part of Th. 2 gives for some $D_k > 0$

$$\Delta_k(x) = O(x^{(2-x_k)/(5-x_k)} \exp (D_k \omega(x))).$$

The special cases of (3.2) and (3.3) when $k = 3$ and $k = 4$ were obtained in [6] and [9].
2. If \( d | n \) and \( (d, n/d) = 1 \), then \( d \) is said to be a unitary divisor of \( n \). For integers \( a, b \) not both zero, let \( (a, b) \) denote the greatest unitary divisor of both \( a \) and \( b \). A divisor \( d > 0 \) of the positive integer \( n \) is called bi-unitary if \( d | n \) and \( (d, n/d) = 1 \). Let \( \tau^{**} (n) \) denote the number of bi-unitary divisors of \( n \). Recently D. Suryanarayana and R. Sitaramachandra Rao proved in [11]

\[
\sum_{n \leq x} \tau^{**} (n) = ax \left( \log x + 2 \gamma - 1 + 2 \sum_p \log p \frac{(p^2 - p - 1)}{(p^4 + 2p^3 + 1)} \right) + E (x),
\]

where \( E (x) = O (x^{1/2} \exp (-A x (x))) \) for some \( A > 0 \), \( \gamma \) is Euler's constant and \( a = \prod_p \frac{1}{1 + \frac{1}{p^2 (p + 1)}} \). If the Riemann hypothesis is true, then for some \( A > 0 \) \( E (x) = O (x^{(2-\varepsilon)/(5-\varepsilon)} \exp (A \omega (x))) \), where \( x_2 (\leq 346/1067) \) is the number appearing in the Dirichlet divisor problem (see (3.1)).

The lengthy proof of (3.4) given in [11] may be shortened as follows. \( \tau^{**} (n) \) is clearly a multiplicative function and

\[
\tau^{**} (p_1^{a_1} \ldots p_r^{a_r}) = \prod_{a_i \text{ even}} a_i \prod_{a_i \text{ odd}} (a_i + 1),
\]

so that for \( \text{Re} \, s > 1 \) we have

\[
\sum_{n=1}^{\infty} \tau^{**} (n) n^{-s} = \prod_p \left( 1 + 2p^{-s} + 2p^{-2s} + 4p^{-3s} + 4p^{-4s} + \ldots \right) = \zeta^2 (s) U (s) / \zeta (2s) = F_2 (s) U (s)
\]

where \( F_2 (s) = \sum_{n=1}^{\infty} f_2 (n) n^{-s} \) is defined at the beginning of this section, and

\( U (s) = \sum_{n=1}^{\infty} u (n) n^{-s} \) is absolutely convergent for \( \text{Re} \, s > 1/3 \). From (3.5) we have

\[
\sum_{n \leq x} \tau^{**} (n) = \sum_{n \leq x} u (n) \sum_{m \leq x/n} f_2 (m), \quad \text{and using } (3.2) \text{ and } (3.3) \text{ for } k = 2 \text{ we obtain (writing } \sum_{n \leq x} u (n) \Delta (x/n) = \sum_{n \leq x} + \sum_{x^{1/2} < n \leq x} \text{ and estimating each sum separately)}
\]

\[
\sum_{n \leq x} \tau^{**} (n) = Ax \log x + Bx + E (x),
\]

where \( E (x) \) is of the form (3.4), and it remains to evaluate \( A \) and \( B \).

Setting \( V (s) = \sum_{n=1}^{\infty} v (n) n^{-s} = U (s) / \zeta (2s) \) we obtain

\[
\sum_{n \leq x} \tau^{**} (n) = \sum_{n \leq x} v (n) \sum_{m \leq x/n} \tau (m) = \sum_{n \leq x} v (n) \left( \frac{x}{n} \log \frac{x}{n} + (2 \gamma - 1) \frac{x}{n} + O (x^{\varepsilon_2} n^{-2}) \right),
\]
where \( \tau(n) \) is the ordinary divisor function. Collecting terms and comparing with (3.6) we get

\[
A = \sum_{n=1}^{\infty} \frac{\nu(n)}{n} = V(1),
\]

\[
B = (2\gamma - 1) V(1) - \sum_{n=1}^{\infty} \frac{\nu(n)}{n} \log n = (2\gamma - 1) V(1) + V'(1).
\]

\[
V(s) = \prod_p (1 - p^{-s})^2 \left(1 + 2p^{-s} + 2p^{-2s} + 4p^{-3s} + 4p^{-4s} + \cdots \right) = \prod_p \left((1 - p^{-s})^2 + 2(p^s + 1)^{-1}\right).
\]

\[
V(1) = \prod_p (1 - 2p^{-1} + p^{-2} + 2/(p + 1)) = a \quad \text{(as given by (3.4))},
\]

\[
V'(s)/V(s) = (\log V(s))' = 2\sum_p \log p \cdot \frac{(1 - p^{-s})p^{-s} - p^{s}(p^s + 1)^{-1}}{(1 - p^{-s})^2 + 2(p^s + 1)^{-1}}.
\]

Therefore \( V'(1) = 2 V(1) \sum_p \frac{p^2 - p^{-1}}{p^4 + 2p^3 + 2p + 1} \log p \), which shows that \( A \) and \( B \)
have the same values as the corresponding constants in (3.4).

3. In [2] E. Cohen defined an exponentially odd integer as an integer \( n = p_1^{a_1} \cdots p_k^{a_k} \) where \( a_1, \ldots, a_k \) are odd numbers and proved

\[
Q^*(x) - \prod_p (1 - (p^2 + p)^{-1}) \cdot x + O(x^{1/2} \log x),
\]

where \( Q^*(x) \) is the number of exponentially odd integers not exceeding \( x \). If we set

\[
h(n) = \begin{cases} 
1 & \text{if } n \text{ is exponentially odd} \\
0 & \text{otherwise}
\end{cases}, \quad \text{then } \sum_{n \leq x} h(n) \text{ and}
\]

\[
H(s) = \sum_{n=1}^{\infty} h(n) n^{-s} = \prod_p (1 + p^{-s} + p^{-3s} + p^{-5s} + \cdots) = \prod_p (1 + p^{-s} (1 - p^{-2s})^{-1}) = F(s)/\zeta(2s)
\]

where \( F(s) = \sum_{n=1}^{\infty} f(n) n^{-s} = \zeta(s) G(s) \) and \( G(s) \) is absolutely convergent for \( Re s > 1/3 \), so that for every \( \varepsilon > 0 \)

\[
\sum_{n \leq x} f(n) = G(1) x + O(x^{1/3 + \varepsilon}).
\]
Since \( h(n) = \sum_{d \mid n} \mu(d) f(n/d^2) \) Theorem 2 sharpens (3.7) to
\[ Q^*(x) = \prod_{p} \left( 1 - (p^2 + p)^{-1} \right) \cdot x + O\left( x^{1/2} \exp \left( -C \varepsilon(x) \right) \right), \quad C > 0. \]

4. Theorem 2 may be also applied to the functions generated by \( \zeta(as) / \zeta(cs) \), where \( a, b, c \) are natural numbers and \( c \geq 2 \). If we write
\[ \sum_{n=1}^{\infty} d(a, b; n) n^{-s} = \zeta(as) \zeta(bs) \] then for \( a \neq b \)
\[ \sum_{n \leq x} d(a, b; n) = \sum_{\substack{m^a \cdot n^b, \leq x}} 1 = \zeta(b/a) x^{1/a} + \zeta(a/b) x^{1/b} + \Delta(x; a, b). \]

H. - E. Richert proved in [7] that for \( b > a \geq 1 \) the following estimates hold:
\( \Delta(x; a, b) = O\left( x^{2/(a + 3b)} \right) \) if \( b < 2a \), \( \Delta(x; a, b) = O\left( x^{2/9a} \log x \right) \) if \( b = 2a \), \( \Delta(x; a, b) = = O\left( x^{2/(b + 5c)} \right) \) if \( b > 2a \). Therefore for example
\[ \sum_{m^a \cdot n^b, \leq x} 1 = \zeta(3/2) x^{1/2} + \zeta(2/3) x^{1/3} + O\left( x^a \right) \]
where certainly \( a \leq 2/15 \), and if we set \( \sum_{n=1}^{\infty} a_k(n) n^{-s} = \zeta(2s) \zeta(3s) / \zeta(k s) \) then for \( 4 \leq k \leq 7 \) by Theorem 2 we obtain
\[ \sum_{n \leq x} a_k(n) = \left( \zeta(3/2) / \zeta(2/3) \right) x^{1/2} + \left( \zeta(2/3) / \zeta(1/3) \right) x^{1/3} + \Delta_k(x) \]
where for some \( C = C(k) > 0 \) we have \( \Delta_k(x) = O\left( x^{1/k} \exp \left( -C \varepsilon(x) \right) \right) \), and if the Riemann hypothesis is true, then \( \Delta_k(x) = O\left( x^{(1-a)/(1+2-k\alpha)} \exp \left( D \omega(x) \right) \right) \) for some \( D = D(k) > 0 \). It can be seen that \( \sum_{n \leq x} a_4(n) \) represents the number of \( n \) not exceeding \( x \) of the form \( n = p_1^{a_1} \ldots p_i^{a_i} \), where every \( a_j (j = 1, \ldots, i) \) is of the form \( 3m \) or \( 3m - 1 \), \( \sum_{n \leq x} a_5(n) = \sum_{n^2 \cdot n^3 = x, (n, m) = 1} 1 \), and \( \sum_{n \leq x} a_6(n) \) represents the number of \( n \) not exceeding \( x \) of the form \( n = p_1^{a_1} \ldots p_i^{a_i} \), where \( a_i \geq 2, \ldots, a_i \geq 2 \). Such numbers are called powerful or squarefull numbers, and the estimate (3.9) for the case \( k = 6 \) was obtained by D. Suryanarayana and R. Sitaramachandra Rao in [10].

5. If in the previous example \( c = a + b \) then
\[ \zeta(as) / \zeta((a + b)/s) = \sum_{n=1}^{\infty} \tau_{a,b}(n) n^{-s} = \sum_{a \cdot b \cdot n, (a, b) = 1} 1. \]

Using (3.8) and Theorem 2 we obtain \( b > a \geq 1 \)
\[ \sum_{n \leq x} \tau_{a,b}(n) = \left[ \zeta((b/a)/s) ((a + b)/a) \right] x^{1/a} + \left[ \zeta((a/b)/s) ((a + b)/b) \right] x^{1/b} + \]
\[ + O\left( x^{a+b} \exp \left( -C \varepsilon(x) \right) \right). \]
where $C$ is a positive constant depending on $a$ and $b$. The formula (3.10) was obtained by E. Cohen in [3] with the poorer error term $O \left( x^{1/(a+b)} \log x \right)$. Cohen considered the integers $n=p_i^{a_i} \ldots p_i^{a_i}$ and denoted by $S_{a,b}$ the set of integers $n$ such that every $a_j (j=1, \ldots, i)$ is divisible by either $a$ or $b$, and by $S_{a,b}^*$ the set of integers such that every $a_j$ is divisible by either $a$ or $b$ but not by both. If we set

$$j_{a,b}(n) = \begin{cases} 1 & n \in S_{a,b}^* \\ 0 & n \notin S_{a,b}^* \end{cases}, \quad j_{a,b}^*(n) = \begin{cases} 1 & n \in S_{a,b} \\ 0 & n \notin S_{a,b} \end{cases}$$

then $S_{a,b}(x) = \sum_{n \leq x} j_{a,b}(n)$ and $S_{a,b}^*(x) = \sum_{n \leq x} j_{a,b}^*(n)$ represent the number of integers from $S_{a,b}$ and $S_{a,b}^*$ respectively not exceeding $x$, and Cohen obtained

(3.11) $S_{a,b}(x) = Ax^{1/a} + Bx^{1/b} + O \left( x^{1/(a+b)} \log x \right)$,

(3.12) $S_{a,b}^*(x) = A^* x^{1/a} + B^* x^{1/b} + O \left( x^{1/(a+b)} \log x \right)$,

where $A, B, A^*, B^*$ are explicit constants depending on $a$ and $b$, $(a, b) = 1$, $b > a > 1$.

Since $j_{a,b}(n)$ is multiplicative and $j_{a,b}^*(n) = \begin{cases} 1 & \text{if } a|n \text{ or } b|n \\ 0 & \text{otherwise} \end{cases}$,

$$J_{a,b}(s) = \sum_{n=1}^{\infty} j_{a,b}(n) n^{-s} - F_{a,b}(s) H_{a,b}(s)$$

where $F_{a,b}(s) = \zeta (as) \zeta (bs)/\zeta ((a+b)s)$ and $H_{a,b}(s) = \prod_p \left( 1 - \frac{(p^a s - 1)(p^b s - 1)}{(p^{(a+b)s} - 1)} \right)$ has the abscissa of absolute convergence equal to $1/ab$. Likewise since $j_{a,b}^*(n)$ is also multiplicative

$$J_{a,b}^*(s) = \sum_{n=1}^{\infty} j_{a,b}^*(n) n^{-s} = F_{a,b}(s) H_{a,b}^*(s),$$

$$H_{a,b}^*(s) = \prod_p \left( 1 - 2 \frac{(p^a s - 1)(p^b s - 1)}{(p^{(a+b)s} - 1)} \right).$$

Using (3.8) and Theorem 2. we obtain $(b > a > 1), (a, b) = 1$

(3.13) $\sum_{n \leq x} f_{a,b}(n) = \zeta (b/a) H_{a,b} \left( 1/a \right) x^{1/a} + \zeta (a/b) H_{a,b} \left( 1/b \right) x^{1/b} + O \left( x^{D(a,b)} \right)$,

(3.14) $\sum_{n \leq x} f_{a,b}^*(n) = \zeta (b/a) H_{a,b}^* \left( 1/a \right) x^{1/a} + \zeta (a/b) H_{a,b}^* \left( 1/b \right) x^{1/b} + O \left( x^{E(a,b)} \right)$

where $D(a, b) < 1/(a+b)$ and $E(a, b) < 1/(a+b)$ and

$$\sum_{n=1}^{\infty} f_{a,b}(n) n^{-s} = \zeta (as) \zeta (bs) H_{a,b}(s), \quad \sum_{n=1}^{\infty} f_{a,b}^*(n) n^{-s} = \zeta (as) \zeta (bs) H_{a,b}^*(s).$$

Since $j_{a,b}(n) = \sum_{d^a+b|n} \mu(d) f_{a,b}(n/d^a+b)$ and $j_{a,b}^*(n) = \sum_{d^a+b|n} \mu(d) f_{a,b}^*(n/d^a+b)$
Theorem 2 gives \( b \gg a \gg 1 \), \((a, b) = 1\) the following improvement of (3.11) and (3.12):

\[
S_{a,b}(x) = Ax^{1/a} + Bx^{1/b} + O(x^{1/(a+b)} \exp(-C \varepsilon(x))),
\]

\[
S_{a,b}^*(x) = A^* x^{1/a} + B^* x^{1/b} + O(x^{1/(a+b)} \exp(-D \varepsilon(x))),
\]

where \( C \) and \( D \) are positive constants depending on \( a \) and \( b \) and

\[
A = \left( \zeta(b/a) H_{a,b} (1/a) / \zeta((a+b)/a) \right), \quad A^* = \left( \zeta(b/a) H_{a,b}^* (1/a) / \zeta((a+b)/a) \right),
\]

\[
B = \left( \zeta(a/b) H_{a,b} (1/b) / \zeta((a+b)/b) \right), \quad B^* = \left( \zeta(a/b) H_{a,b}^* (1/b) / \zeta((a+b)/b) \right).
\]

The constants \( A^* \) and \( B^* \) obtained by E. Cohen in [3] differ from the above ones and are incorrect, due to a mistake in his equation (4.14).

**REFERENCES**


Aleksandar Ivić
Rudarsko-geološki fakultet
Džušina 7, 11000 Beograd
Yugoslavia

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