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A CONVOLUTION THEOREM WITH APPLICATIONS TO SOME DIVISOR FUNCTIONS

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1. Introduction

The convolution of two arithmetical functions $f(n)$ and $g(n)$ (or Dirichlet convolution, to distinguish it from unitary and other possible arithmetical convolutions) is the function

$$h(n)=\sum_{d\mid n} f(n/d)\,g(d)=\sum_{d\mid n} g(n/d)\,f(d), \tag{1.1}$$

where the sum is taken over all positive divisors of n. A common procedure in dealing with the asymptotic formula for the sum $\sum_{n\leq x} h(n)$ is to express $h(n)$ as a convolution of $f(n)$ and $g(n)$ and to derive the asymptotic formula for $\sum_{n\leq x} h(n)$ from the asymptotic formulas for $\sum_{n\leq x} f(n)$ and $\sum_{n\leq x} g(n)$. Such convolution methods were investigated by many authors, and notably by J. P. Tull who in [13] and [14] proved two theorems for the even more general case of the Stieltjes convolution $\int_1^x A(x/u)\,dB(u)$.

This paper contains two convolution theorems with sharp error terms, of which Theorem 1 is very general, while Theorem 2 may be regarded as a special case of Theorem 1 when $g(n)=\mu(n)$. Theorem 2 gives also the error term under the assumption that the famous Riemann hypothesis about the non-trivial zeros of the zeta function is true.

In the formulation of both theorems instead of the convolution (1.1) we use

$$h(n)=\sum_{d^k\mid n} f(n/d^k)\,g(d). \tag{1.2}$$

which may be reduced at once to the form (1.1) by setting (k is a fixed integer)

$$G(n)=\begin{cases} g(m) & n=m^k \\ 0 & n\neq m^k. \end{cases} \tag{1.3}$$

The reason for introducing (1.2) lies in the nature of applications of Theorem 2, since many divisor functions $h(n)$ may be expressed as $h(n)=\sum_{d^k|n}\mu(d)f(n/d^k)$, so that Theorem 2 is readily applicable. A number of these applications is given in Section 3.

For the more general Theorem 1 some properties of slowly oscillating functions are needed. By a slowly oscillating (also called slowly varying) function we shall mean a positive function $L(x)$ defined for $x>0$ and continuous for $x\geqslant x_0>0$, such that for every $c>0$

$$\lim_{x\to\infty} L(cx)/L(x)=1. \tag{1.4}$$

J. Karamata in [4] characterized such functions in the form

$$L(x)=a(x)\exp\Big(\int_{x_0}^{x}\delta(t)\,t^{-1}\,dt\Big) \tag{1.5}$$

where $a(x)$ and $\delta(x)$ are continuous for $x\geqslant x_0$, $a(x)\to a_0>0$ and $\delta(x)\to 0$ as $x\to\infty$. Slowly oscillating functions naturally arise in number theory since most of the functions like $\log^A x$, $\log\log x$, $\exp(C\log^B x)$ (for $B<1$) that appear in the asymptotic formulas for arithmetic functions are slowly oscillating. For a comprehensive account of slowly oscillating and the more general slowly varying functions see [8].

2. Statement and proof of theorems

Theorem 1. *Let $f(n)$ be an arithmetical function for which*

$$\sum_{n\leq x} f(n)=\sum_{i=1}^{l} c_i x^{a_i}L_i(x)+O(x^a),\quad \sum_{n\leq x}|f(n)|=O(x^{a_1}P(x)), \tag{2.1}$$

where $a_1\geqslant a_2\geqslant\cdots\geqslant a_l>1/k>a\geqslant 0$, $c_1,\ldots,c_l$ are constants, k is a fixed natural number, $L_1(x),\ldots,L_l(x)$ are slowly oscillating functions, and $P(x)$ is a non-decreasing slowly oscillating function. Let further $g(n)$ be an arithmetical function for which

$$\sum_{n\leq x} g(n)=O(x^b N(x))\ \textit{for some}\ 0\leqslant b\leqslant 1,\quad \sum_{n\leq x}|g(n))|=O(x), \tag{2.2}$$

$N(x)$ is a slowly oscillating function of the form $N(x)=\exp(C\omega(x))$, $\omega(x)=\int_{x_0}^{x}\eta(t)\,t^{-1}\,dt$, $\eta(x)$ is continuous and positive for $x\geqslant x_0$, $\lim_{x\to\infty}\eta(x)=0$,

$\lim_{x\to\infty} P(x)\exp(A\omega(x))=0$ *for every* $A<0$, *and if* $b=1$, C *is negative, and if* $0\leqslant b<1$, C *is positive.*

If $h(n)=\sum_{d^k|n} f(n/d^k)g(d)$ *then there exist functions* $Q_1(x),\ldots,Q_l(x)$ *such that* $Q_i(x)=O(x^\varepsilon)$ *for every* $\varepsilon>0$ *and* $i=1,\ldots,l$ *and*

$$\sum_{n\leq x} h(n)=\sum_{i=1}^{l} c_i x^{a_i} Q_i(x)+\Delta(x), \tag{2.3}$$

where in the case $b=1$ $\Delta(x)=O(x^{1/k}\exp(D\omega_1(x)))$, $\omega_1(x)=\int_{x_1}^{x}\eta(t^u)t^{-1}\,dt$ *with* $x_1=x_0^{1/u}$, $D<0$ *for every* $u<1/k$. *In the case* $0\leqslant b<1$ *we have* $\Delta(x)=O(x^c\exp(D\omega(x)))$, *where* $D>0$ *and* $c=(a_1-ab)/(a_1a-ak+1-b)$.

Proof. Let $y, z>1$ and $yz=x$. Then using (1.3) we get

$$\sum_{n\leq x} h(n)=\sum_{n\leq x}\sum_{d|n} G(d)f(n/d)=\sum_{mn\leq x} G(m)f(n)=$$

$$\sum_{m\leq y} G(m)\sum_{n\leq x/m} f(n)+\sum_{n\leq z} f(n)\sum_{m\leq x/n} G(m)-\sum_{m\leq y} G(m)\sum_{n\leq z} f(n)=S_1+S_2-S_3.$$

$$\sum_{n\leq x} G(n)=\sum_{n\leq x^{1/k}} g(n)=O(x^{b/k}N(x^{1/k})),\quad \sum_{n\leq x}|G(n)|=\sum_{n\leq x^{1/k}}|g(n)|=O(x^{1/k}),$$

so that we obtain

$$S_1=\sum_{m\leq y} G(m)\sum_{n\leq x/m} f(n)=\sum_{i=1}^{l} c_i x^{a_i}\sum_{m\leq y} G(m)\,m^{-a_i}L_i(x/m)+O\Big(x^a\sum_{m\leq y}|G(m)|m^{-a}\Big)=$$

$$\sum_{i=1}^{l} c_i x^{a_i} Q_i(x)+O(x^a y^{1/k-a}),$$

where $y=y(x)$ will be suitably chosen later, and where we have set

$$Q_i(x)=\sum_{m\leq y} G(m)\,m^{-a_i}L_i(x/m)=\sum_{m\leq y^{1/k}} g(m)\,m^{-ka_i}L_i(x/m^k). \tag{2.4}$$

i) **The case** $b=1$. If $b=1$ then $N(x)$ it decreasing and therefore for $n\leqslant z$ we have $N((x/n)^{1/k})\leqslant N(y^{1/k})$ which gives

$$S_2=\sum_{n\leq z} f(n)\sum_{m\leq x/n} G(m)=O(x^{1/k}N(y^{1/k})\sum_{n\leq z}|f(n)|n^{-1/k})=$$

$$O(x^{1/k}z^{a_1-1/k}P(z)N(y^{1/k}))=O(x^{a_1}y^{1/k-a_1}P(x/y)N(y^{1/k})).$$

$$S_3=\sum_{m\leq y} G(m)\sum_{n\leq z} f(n)=O(z^{a_1}P(z)y^{1/k}N(y^{1/k}))=O(x^{a_1}y^{1/k-a_1}P(x/y)N(y^{1/k})).$$

Therefore we obtain

$$\sum_{n\le x} h(n)=\sum_{i=1}^{l} c_i x^{a_i} Q_i(x)+O(x^a y^{1/k-a})+O(x^{a_1} y^{1/k-a_1} P(x/y) N(y^{1/k})). \tag{2.5}$$

Let now $0<u<1/k$ and choose $y=x(N(x^u))^{1/(a_1-a)}$, so that $y<x$ for $x>x_0$. From (1.5) it follows that $L(x)=O(x^\varepsilon)$ for every $\varepsilon>0$ if $L(x)$ is slowly oscillating, which gives $x^u\leqslant x^{1/k-\varepsilon}\leqslant y^{1/k}$ for $0<\varepsilon<1/k-u$, so that $N(y^{1/k})\leqslant \leqslant N(x^u)$. This means that the error terms in (2.5) may be written as

$$O\left(x^{1/k}(N(x^u))^{(1/k-a)/(a_1-a)}\left(1+\frac{N(y^{1/k})}{N(x^u)}P(x/y)\right)\right)=$$

$$O(x^{1/k}(N(x^u))^{(1/k-a)/(a_1-a)}P(x^u)),$$

since $x/y<x^u$ for x large enough. If $C_1=(C/k-Ca)/(a_1-a)$, then for every $A<0$

$$(N(x^u))^{(1/k-a)/(a_1-a)}P(x^u)=P(x^u)\exp(A\,\omega(x^u))\exp((C_1-A)\,\omega(x^u))=$$

$$=O(\exp(D\,\omega_1(x))),$$

where $D=(C_1-A)u$, $\omega_1(x)=\int\limits_{x_1}^{x}\eta(t^u)\,t^{-1}\,dt$, $x_1=x_0^{1/u}$, since

$$\lim_{x\to\infty} P(x^u)\exp(A\,\omega(x^u))=0.$$

ii) **The case $0\leqslant b<1$.**. If $0\leqslant b<1$ then $N(x)$ is increasing and therefore $N(x^{1/k}n^{-1/k})\leqslant N(x)$, so that

$$S_2=O(x^{b/k}N(x)\sum_{n\le z}|f(n)|\,n^{-b/k})=O(x^{b/k}N(x)\,P(x)\,z^{a_1-b/k})=$$

$$O(x^{a_1}y^{b/k-b_1}N(x)\,P(x)),$$

and the same estimate holds for S_3, which yields

$$\sum_{n\le x} h(n)=\sum_{i=1}^{l} c_i x^{a_i} Q_i(x)+O(x^a y^{1/k-a})+O(x^{a_1} y^{b/k-a_1} N(x)\,P(x)). \tag{2.6}$$

If $D>C$ then

$$N(x)\,P(x)=\exp(D\,\omega(x))\exp((C-D)\,\omega(x))\,P(x)=O(\exp(D\,\omega(x))),$$

since $\lim\limits_{x\to\infty} P(x)\exp(A\,\omega(x))=0$ for $A=C-D<0$. Taking now $y=x^q$ where $q=k(a_1-a)/(1-b+k(a_1-a))$ we obtain finally

$$\sum_{n\le x} h(n)=\sum_{i=1}^{l} c_i x^{a_i} Q_i(x)+O(x^c\exp(D\,\omega(x))),$$

where $c=(a_1-ab)/(1-b+k(a_1-a))$, as stated in the theorem.

Concerning the functions $Q_i(x)$ it follows from (2.4)

$$Q_i(x) = O\big(\sum_{m\leq y} |G(m)|\, m^{-a_i} L_i(x/m)\big) = O\big(x^{\varepsilon} \sum_{m\leq y} |G(m)|\, m^{-a_i-\varepsilon}\big) = O(x^{\varepsilon}),$$

since $L_i(x) = O(x^{\varepsilon})$, and the second sum above is bounded. It may be further shown that

$$\lim_{x\to\infty} Q_i(x)/L_i(x) = \sum_{n=1}^{\infty} g(n)\, n^{-ka_i}, \tag{2.7}$$

which means that $Q_i(x)$ is slowly oscillating if it is continuous and the above limit is positive, since it is then asymptotic to a slowly oscillating function. A more detailed discussion is omitted, since in many applications to divisor problems the functions $Q_i(x)$ turn out to be polynomials in $\log x$.

Theorem 2. *Let $f(n)$ be an arithmetical function for which*

$$\sum_{n\leq x} f(n) = \sum_{i=1}^{l} x^{a_i} P_i(\log x) + O(x^{a}), \quad \sum_{n\leq x} |f(n)| = O(x^{a_1} \log^r x) \tag{2.8}$$

where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_l > 1/k > a \geqslant 0$, $r \geqslant 0$, $P_1(t), \ldots, P_l(t)$ are polynomials in t with degrees not exceeding r, and k is a fixed natural number.

If $h(n) = \sum_{d^k | n} \mu(d) f(n/d^k)$ where $\mu(n)$ is the Möbius function, then

$$\sum_{n\leq x} h(n) = \sum_{i=1}^{l} x^{a_i} R_i(\log x) + \Delta(x), \tag{2.9}$$

where $R_1(t), \ldots, R_l(t)$ are polynomials in t, and for some $D>0$

$$\Delta(x) = O(x^{1/k} \exp(-D \log^{3/5} x \cdot (\log\log x)^{-1/5})). \tag{2.10}$$

If the Riemann hypothesis is true, then for some $D>0$

$$\Delta(x) = O(x^{c} \exp(D \log x \cdot (\log\log x)^{-1})),\quad c = (2a_1 - a)/(2ka_1 - 2ka + 1). \tag{2.11}$$

Proof. Theorem 2 is a special case of Theorem 1 when $g(n) = \mu(n)$, $c_i L_i(x) = P_i(\log x)$, $P(x) = \log^r x$. For $\sum_{n\leq x} \mu(n)$ we use the following best-known estimate due to A. Walfisz [15]:

$$M(x) = \sum_{n\leq x} \mu(n) = O(x \exp(-C\varepsilon(x))), \tag{2.12}$$

where $C>0$ and from now on $\varepsilon(x)$ denotes $\varepsilon(x) = \log^{3/5} x \cdot (\log\log x)^{-1/5}$. This corresponds to the case $b=1$ of Th. 1; if the Riemann hypothesis that all nontrivial zeros of $\zeta(s)$ lie on $s = \frac{1}{2} + it$ is true, then as shown in [12], Ch. XIV

$$M(x) = \sum_{n\leq x} \mu(n) = O(x^{1/2} \exp(C\omega(x))), \tag{2.13}$$

where $C>0$ and from now on $\omega(x)$ denotes $\omega(x)=\log x\cdot(\log\log x)^{-1}$, and this corresponds to the case $b=\frac{1}{2}$ of Th. 1. If one could prove for some $1/2<b<1$ $M(x)=O(x^b)$, then Th. 1 would give for some $s\geq 0$ $\Delta(x)=$ $=O(x^c\log^s x)$, where $c=(a_1-ab)/(a_1-ak+1-b)$. It should be noted that

$$c_i\sum_{m\leq y}G(m)m^{-a_i}L_i(x/m)=c_i\sum_{m=1}^{\infty}\mu(m)m^{-ka_i}L_i(x/m^k)-$$

$$-c_i\sum_{m^k>y}\mu(m)m^{-ka_i}L_i(x/m^k),$$

and that $L_i(x/m^k)$ can be written as a polynomial in $\log x$, so that

$$c_i\sum_{m=1}^{\infty}\mu(m)m^{-ka_i}L_i(x/m^k)=R_i(\log x),$$

where $R_i(t)$ is a polynomial in t, and it remains to show that sums of the type $\sum_{m>y^{1/k}}\mu(m)m^{-ka_i}\log^A m$ contribute to the error term. If we set $y^{1/k}=v$, $ka_i=c>1$, then

$$\sum_{m>v}\mu(m)m^{-c}\log^A m=\int_v^{\infty}t^{-c}\log^A t\cdot dM(t)=$$

$$=v^{-c}M(v)\log^A v+O\left(\int_v^{\infty}|M(t)|t^{-c-1}\log^A t\cdot dt\right).$$

If we use (2.12) then $\exp(-C\varepsilon(x))$ is decreasing for $x\geq x_1$ and thus

$$\sum_{m>v}\mu(m)m^{-c}\log^A m=O(v^{1-c}\exp(-C\varepsilon(v))\log^A v)+$$

$$+O(\exp(-C\varepsilon(v))\int_v^{\infty}t^{-c}\log^A t\cdot dt)=O(v^{1-c}\exp(C\varepsilon(v))\log^A v).$$

If we use (2.13) then $x^{-1/2}\exp(C\omega(x))$ is decreasing for $x\geq x_2$ and so

$$\sum_{m>v}\mu(m)m^{-c}\log^A m=O(v^{1/2-c}\exp(C\omega(v))\log^A v)+$$

$$+O(v^{-1/2}\exp(C\omega(v))\int_v^{\infty}t^{-c}\log^A t dt)=O(v^{1/2-c}\exp(C\omega(v))\log^A v).$$

The remaining details of the proof are the same as in Theorem 1; note that

$$\omega_1(x)\sim u^{-2/5}\varepsilon(x).$$

3. Applications

1. Let first $F_k(s)=\sum_{n=1}^{\infty} f_k(n) n^{-s}=\zeta^k(s)/\zeta(2s)$ for $k\geqslant 2$. Then we have $f_k(n)=\sum_{d^2|n}\mu(d)\tau_k(n/d^2)$ where $\tau_k(n)$ is the number of representations of n as a product of k factors and $\sum_{n=1}^{\infty}\tau_k(n)n^{-s}=\zeta^k(s)$. Since $\tau_k(n)$ is multiplicative and $\tau_k(p^a)=\binom{a+k-1}{k-1}$ then $f_k(n)$ is also multiplicative and

$$f_k(p^a)=\binom{a+k-1}{k-1}-\binom{a+k-3}{k-1}=\frac{2a+k-2}{a}\binom{a+k-3}{k-2}.$$

For special values of k this gives the following well-known arithmetical functions:

a) for $k=2$ we have $f_2(p^a)=2$ so that $f_2(n)=\sum_{d|n,\,(d,n/d)=1} 1$,

b) for $k=3$ we have $f_3(p^a)=2a+1$ so that $f_3(n)=\tau(n^2)$, where $\tau(n)$ is the number of divisors of n,

c) for $k=4$ we have $f_4(p^a)=(a+1)^2$ so that $f_4(n)=\tau^2(n)$.

It is known (see [12], Ch. XII) that

$$\sum_{n\leq x}\tau_k(n)=xP_{k-1}(\log x)+O(x^{\alpha_k}). \tag{3.1}$$

where $P_{k-1}(t)$ is a polynomial of degree $k-1$ in t, $\alpha_k>(k-1)/2k$, for $k\geqslant 4$ and every $\varepsilon>0$ $\alpha_k\leqslant(k-1)/(k+2)+\varepsilon$, $\alpha_2\leqslant 346/1067$, $\alpha_3\leqslant 5/11$ (see [5] and [4]). It has been conjectured (see [12], Ch. XII) that for every $\varepsilon>0$ and $k\geqslant 2$ one has $\alpha_k=(k-1)/2k+\varepsilon$. If we suppose that $\alpha_k<1/2$ (so far this has been shown to be true only when $k=2$ and $k=3$), then Th. 2 may be applied at once to give

$$\sum_{n\leq x} f_k(n)=xH_{k-1}(\log x)+\Delta_k(x),\ \Delta_k(x)=O(x^{1/2}\exp(-C_k\,\varepsilon(x))), \tag{3.2}$$

where $H_{k-1}(t)$ is a polynomial of degree $k-1$ in t whose coefficients may be found for example by residues, and $C_k>0$ is a constant depending on k.

If besides $\alpha_k<1/2$ we assume the truth of the Riemann hypothesis, then the second part of Th. 2 gives for some $D_k>0$

$$\Delta_k(x)=O(x^{(2-\alpha_k)/(5-\alpha_k)}\exp(D_k\,\omega(x))). \tag{3.3}$$

The special cases of (3.2) and (3.3) when $k=3$ and $k=4$ were obtained in [6] and [9].

2. If $d \mid n$ and $(d, n/d)=1$, then d is said to be a unitary divisor of n. For integers a, b not both zero, let $(a, b)^{**}$ denote the greatest unitary divisor of both a and b. A divisor $d>0$ of the positive integer n is called bi-unitary if $d \mid n$ and $(d, n/d)^{**}=1$. Let $\tau^{**}(n)$ denote the number of bi-unitary divisors of n Recently D. Suryanarayana and R. Sitaramachandra Rao proved in [11]

$$(3.4) \qquad \sum_{n\le x} \tau^{**}(n) = ax\left((\log x + 2\gamma - 1 + 2\sum_p \log p \frac{(p^2-p-1)}{(p^4+2p^3+1)}\right) + E(x),$$

where $E(x) = O(x^{1/2}\exp(-A\varepsilon(x)))$ for some $A>0$, γ is Euler's constant and $a=\prod_p (1-(p-1)/p^2(p+1)))$. If the Riemann hypothesis is true, then for some $A>0$ $E(x)=O(x^{(2-\alpha_2)/(5-\alpha_2)}\exp(A\omega(x)))$, where $\alpha_2\ (\leqslant 346/1067)$ is the number appearing in the Dirichlet divisor problem (see (3.1)).

The lengthy proof of (3.4) given in [11] may be shortened as follows. $\tau^{**}(n)$ is clearly a multiplicative function and

$$\tau^{**}(p_1^{a_1}\dots p_r^{a_r}) = \prod_{a_i\ even} a_i \prod_{a_i\ odd} (a_i+1),$$

so that for Re $s>1$ we have

$$(3.5) \qquad \sum_{n=1}^{\infty} \tau^{**}(n) n^{-s} = \prod_p (1+2p^{-s}+2p^{-2s}+4p^{-3s}+4p^{-4s}+\dots) =$$

$$= \zeta^2(s)U(s)/\zeta(2s) = F_2(s)U(s)$$

where $F_2(s)=\sum_{n=1}^{\infty} f_2(n)n^{-s}$ is defined at the beginning of this section, and $U(s)=\sum_{n=1}^{\infty} u(n)n^{-s}$ is abolutely convergent for $Re\, s>1/3$. From (3.5) we have $\sum_{n\le x}\tau^{**}(n)=\sum_{n\le x}u(n)\sum_{m\le x/n} f_2(m)$, and using (3.2) and (3.3) for $k=2$ we obtain (writing $\sum_{n\le x} u(n)\Delta(x/n) = \sum_{n\le x^{1/2}} + \sum_{x^{1/2}<n\le x}$ and estimating each sum separately)

$$(3.6) \qquad \sum_{n\le x} \tau^{**}(n) = Ax\log x + Bx + E(x),$$

where $E(x)$ is of the form (3.4), and it remains to evaluate A and B.

Setting $V(s)=\sum_{n=1}^{\infty} v(n)n^{-s} = U(s)/\zeta(2s)$ we obtain

$$\sum_{n\le x}\tau^{**}(n) = \sum_{n\le x} v(n)\sum_{m\le x/n}\tau(m) = \sum_{n\le x} v(n)\left(\frac{x}{n}\log\frac{x}{n} + (2\gamma-1)\frac{x}{n} + O(x^{\alpha_2}n^{-\alpha_2})\right),$$

where $\tau(n)$ is the ordinary divisor function. Collecting terms and comparing with (3.6) we get

$$A=\sum_{n=1}^{\infty} v(n)/n=V(1),$$

$$B=(2\gamma-1)V(1)-\sum_{n=1}^{\infty} v(n)n^{-1}\log n=(2\gamma-1)V(1)+V'(1).$$

$$V(s)=\prod_p (1-p^{-s})^2(1+2p^{-s}+2p^{-2s}+4p^{-3s}+4p^{-4s}+\cdots)=$$

$$=\prod_p ((1-p^{-s})^2+2(p^s+1)^{-1}).$$

$$V(1)=\prod_p (1-2p^{-1}+p^{-2}+2/(p+1))=a \text{ (as given by (3.4))},$$

$$V'(s)/V(s)=(\log V(s))'=2\sum_p \log p\cdot\frac{(1-p^{-s})p^{-s}-p^s(p^s+1)^{-2}}{(1-p^{-s})^2+2(p^s+1)^{-1}}.$$

Therefore $V'(1)=2V(1)\sum_p \frac{p^2-p-1}{p^4+2p^3+1}\log p$, which shows that A and B have the same values as the corresponding constants in (3.4).

3. In [2] E. Cohen defined an exponentially odd integer as an integer $n=p_1^{a_1}\ldots p_1^{a_i}$ where $a_1,\ldots,a_i$ are odd numbers and proved

$$Q^*(x)=\prod_p (1-(p^2+p)^{-1})\cdot x+O(x^{1/2}\log x), \tag{3.7}$$

where $Q^*(x)$ is the number of exponentially odd integers not exceeding x. If we set

$$h(n)=\begin{cases}1 & \text{if } n \text{ is exponentially odd}\\ 0 & \text{otherwise}\end{cases},\quad \text{then } Q^*(x)=\sum_{n\le x} h(n) \text{ and}$$

$$H(s)=\sum_{n=1}^{\infty} h(n)n^{-s}=\prod_p (1+p^{-s}+p^{-3s}+p^{-5s}+\cdots)=$$

$$=\prod_p (1+p^{-s}(1-p^{-2s})^{-1})=F(s)/\zeta(2s)$$

where $F(s)=\sum_{n=1}^{\infty} f(n)n^{-s}=\zeta(s)G(s)$ and $G(s)$ is absolutely convergent for $Re\, s>1/3$, so that for every $\varepsilon>0$

$$\sum_{n\le x} f(n)=G(1)x+O(x^{1/3+\varepsilon}).$$

Since $h(n)=\sum_{d^2 \mid n} \mu(d) f(n/d^2)$ Theorem 2 sharpens (3.7) to

$$Q^*(x)=\prod_p (1-(p^2+p)^{-1})\cdot x+O(x^{1/2}\exp(-C\varepsilon(x))),\ C>0.$$

4. Theorem 2 may be also applied to the functions generated by $\zeta(as)\zeta(bs)/\zeta(cs)$, where a, b, c are natural numbers and $c\geqslant 2$. If we write

$$\sum_{n=1}^{\infty} d(a, b; n) n^{-s}=\zeta(as)\zeta(bs) \text{ then for } a\neq b$$

(3.8) $$\sum_{n\leq x} d(a, b; n)=\sum_{m^a n^b\leq x} 1=\zeta(b/a)x^{1/a}+\zeta(a/b)x^{1/b}+\Delta(x; a, b).$$

H.-E. Richert proved in [7] that for $b>a\geqslant 1$ the following estimates hold: $\Delta(x; a, b)=O(x^{2/(3a+3b)})$ if $b<2a$, $\Delta(x; a, b)=O(x^{2/9a}\log x)$ if $b=2a$, $\Delta(x; a, b)=O(x^{2/(2b+5c)})$ if $b>2a$. Therefore for example

$$\sum_{m^2 n^3\leq x} 1=\zeta(3/2)x^{1/2}+\zeta(2/3)x^{1/3}+O(x^a)$$

where certainly $a\leqslant 2/15$, and if we set $\sum_{n=1}^{\infty} a_k(n) n^{-s}=\zeta(2s)\zeta(3s)/\zeta(ks)$ then for $4\leqslant k\leqslant 7$ by Theorem 2 we obtain

(3.9) $$\sum_{n\leq x} a_k(n)=(\zeta(3/2)/\zeta(k/2))x^{1/2}+(\zeta(2/3)/\zeta(k/3))x^{1/3}+\Delta_k(x)$$

where for some $C=C(k)>0$ we have $\Delta_k(x)=O(x^{1/k}\exp(-C\varepsilon(x)))$, and if the Riemann hypothesis is true, then $\Delta_k(x)=O(x^{(1-a)/(1+k-2ka)}\exp(D\omega(x)))$ for some $D=D(k)>0$. It can be seen that $\sum_{n\leq x} a_4(n)$ represents the number of n not exceeding x of the form $n=p_1^{a_1}\ldots p_i^{a_i}$, where every a_j $(j=1,\ldots,i)$ is of the form $3m$ or $3m-1$, $\sum_{n\leq x} a_5(n)=\sum_{m^2 n^3\leq x,\ (m,n)=1} 1$, and $\sum_{n\leq x} a_6(n)$ represents the the number of n not exceeding x of the form $n=p_1^{a_1}\ldots p_i^{a_i}$, where $a_1\geqslant 2,\ldots, a_i\geqslant 2$. Such numbers are called powerful or squarefull numbers, and the estimate (3.9) for the case $k=6$ was obtained by D. Suryanarayana and R. Sitaramachandra Rao in [10].

5. If in the previous example $c=a+b$ then

$$\zeta(as)\zeta(bs)/\zeta((a+b)s)=\sum_{n=1}^{\infty} \tau^*_{a,b}(n) n^{-s},\quad \tau^*_{a,b}(n)=\sum_{d^a\delta^b=n,\ (d,\delta)=1} 1.$$

Using (3.8) and Theorem 2 we obtain $(b>a\geqslant 1)$

(3.10) $$\sum_{n\leq x} \tau^*_{a,b}(n)=[\zeta(b/a)/\zeta((a+b)/a)]x^{1/a}+[\zeta(a/b)/\zeta((a+b)/b)]x^{1/b}+$$

$$+O(x^{\frac{1}{a+b}}\exp(-C\varepsilon(x))),$$

where C is a positive constant depending on a and b. The formula (3.10) was obtained by E. Cohen in [3] with the poorer error term $O(x^{1/(a+b)}\log x)$. Cohen considered the integers $n=p_1^{a_1}\dots p_i^{a_i}$ and denoted by $S_{a,b}$ the set of integers n such that every $a_j\,(j=1,\dots,i)$ is divisible by either a or b, and by $S^*_{a,b}$ the set of integers such that every a_j is divisible by either a or b but not by both. If we set

$$j_{a,b}(n)=\begin{cases}1 & n\in S_{a,b}\\ 0 & n\notin S_{a,b}\end{cases},\quad j^*_{a,b}(n)=\begin{cases}1 & n\in S^*_{a,b}\\ 0 & n\notin S^*_{a,b}\end{cases}$$

then $S_{a,b}(x)=\sum_{n\le x} j_{a,b}(n)$ and $S^*_{a,b}(x)=\sum_{n\le x} j^*_{a,b}(n)$ represent the number of integers from $S_{a,b}$ and $S^*_{a,b}$ respectively not exceeding x, and Cohen obtained

$$S_{a,b}(x)=Ax^{1/a}+Bx^{1/b}+O(x^{1/(a+b)}\log x), \tag{3.11}$$

$$S^*_{a,b}(x)=A^*x^{1/a}+B^*x^{1/b}+O(x^{1/(a+b)}\log x), \tag{3.12}$$

where A, B, A^*, B^* are explicit constants depending on a and b, $(a,b)=1$, $b>a>1$.

Since $j_{a,b}(n)$ is multiplicative and $j_{a,b}(p^\alpha)=\begin{cases}1 & \text{if } a\mid\alpha \text{ or } b\mid\alpha\\ 0 & \text{otherwise}\end{cases}$,

$$J_{a,b}(s)=\sum_{n=1}^{\infty} j_{a,b}(n)n^{-s}=F_{a,b}(s)H_{a,b}(s)$$

where $F_{a,b}(s)=\zeta(as)\zeta(bs)/\zeta((a+b)s)$ and $H_{a,b}(s)=\prod_p\left(1-\frac{(p^{as}-1)(p^{bs}-1)}{(p^{abs}-1)(p^{(a+b)s}-1)}\right)$ has the abscissa of absolute convergence equal to $1/ab$. Likewise since $j^*_{a,b}(n)$ is also multiplicative

$$J^*_{a,b}(s)=\sum_{n=1}^{\infty} J^*_{a,b}(n)n^{-s}=F_{a,b}(s)H^*_{a,b}(s),$$

$$H^*_{a,b}(s)=\prod_p\left(1-2\,\frac{(p^{as}-1)(p^{bs}-1)}{(p^{abs}-1)(p^{(a+b)s}-1)}\right).$$

Using (3.8) and Theorem 2. we obtain $(b>a>1,\ (a,b)=1)$

$$\sum_{n\le x} f_{a,b}(n)=\zeta(b/a)H_{a,b}(1/a)x^{1/a}+\zeta(a/b)H_{a,b}(1/b)x^{1/b}+O(x^{D(a,b)}), \tag{3.13}$$

$$\sum_{n\le x} f^*_{a,b}(n)=\zeta(b/a)H^*_{a,b}(1/a)x^{1/a}+\zeta(a/b)H^*_{a,b}(1/b)x^{1/b}+O(x^{E(a,b)}) \tag{3.14}$$

where $D(a,b)<1/(a+b)$ and $E(a,b)<1/(a+b)$ and

$$\sum_{n=1}^{\infty} f_{a,b}(n)n^{-s}=\zeta(as)\zeta(bs)H_{a,b}(s),\quad \sum_{n=1}^{\infty} f^*_{a,b}(n)n^{-s}=\zeta(as)\zeta(bs)H^*_{a,b}(s).$$

Since $j_{a,b}(n)=\sum_{d^{a+b}\mid n}\mu(d)f_{a,b}(n/d^{a+b})$ and $j^*_{a,b}(n)=\sum_{d^{a+b}\mid n}\mu(d)f^*_{a,b}(n/d^{a+b})$

Theorem 2 gives for $b>a>1$, $(a, b)=1$ the following improvement of (3.11) and (3.12):

$$S_{a,b}(x)=Ax^{1/a}+Bx^{1/b}+O(x^{1/(a+b)}\exp(-C\varepsilon(x))), \tag{3.15}$$

$$S^*_{a,b}(x)=A^*x^{1/a}+B^*x^{1/b}+O(x^{1/(a+b)}\exp(-D\varepsilon(x))) \tag{3.16}$$

where C and D are positive constants depending on a and b and

$$A=(\zeta(b/a)H_{a,b}(1/a))/\zeta((a+b)/a),\quad A^*=(\zeta(b/a)H^*_{a,b}(1/a))/\zeta((a+b)/a),$$

$$B=(\zeta(a/b)H_{a,b}(1/b))/\zeta((a+b)/b,\quad B^*=(\zeta(a/b)H^*_{a,b}(1/b))/\zeta((a+b)/b).$$

The constants A^* and B^* obtained by E. Cohen in [3] differ from the above ones and are incorrect, due to a mistake in his equation (4.14).

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