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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN LOCALLY CONVEX SPACES

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In this paper we shall prove some theorems on the fixed point for multivalued mapping $T:M\to Cl(M)$ where M is a closed subset of the sequentially complete locally convex space E and $Cl(M)$ is the family of all closed subsets of the set M. Let $\{p_i\}_{i\in I}$ be the family of seminorms defining the topology in E. First we shall give some notations which we shall use in the sequel.

Let k be a positive integer and for each $i\in\{1, 2, \ldots, k\}=A$ let $f_i: I\to I$. Let $q: IxA\to\mathbf{R}^+$. Also, for each $n\in N$ (positive integers)

$$V(n, k)=\{(J_1, J_2, \ldots, J_n): J_i\in A, i=1, 2, \ldots, n\}$$

For an $a=(J_1, J_2, \ldots, J_n)\in V(n, k)$ and $i\in I$ let:

$$a(i)=f_{J_1}\circ f_{J_2}\circ\cdots\circ f_{J_n}(i)$$

Further, let for $i\in I$ and $n\in N$:

$$Q(i, n)=\max\{q(a(i), r): a\in V(n, k), r\in A\}$$

$$P(i, n, u, x)=\max\{p_{a(i)}(u-x): a\in V(n, k)\}$$

and $Q(i, 0)=\max\{q(i, r): r\in A\}$, $P(i, 0, u, x)=p_i(u-x)$ for every $i\in I$.

Now we shall prove a generalization of Theorem 1 from [1].

Theorem 1. *Let $(E, \{p_i\}_{i\in I})$ be a sequentially complete locally convex space, $T: M\to Cl(M)$, where M is a closed subset of E such that the following conditions are satisfied:*

1. *For every $u, v\in M$ and every $x\in Tu$ there exists $y\in Tv$ such that:*

$$p_i(x-y)\leqslant\sum_{r=1}^{k} q(i, r)\, p_{f_r(i)}(u-v) \text{ for every } i\in I$$

2. *There exist* $x_0 \in M$ *and* $x_1 \in Tx_0$ *so that:*

$$R = \sup_{i \in I} \overline{\lim_{n \in N}} \sqrt[n]{P(i, n, x_1, x_0) \prod_{r=0}^{n-1} Q(i, r)} < \frac{1}{k}$$

Then there exists at least one element $x \in M$ *such that* $x \in Tx$.

Proof; Let x_0 and x_1 be elements given by 2. It follows by 1. that there exists a sequence $\{x_n\}_{n \in N} \subseteq M$ such that:

(i) $x_{a+1} \in Tx_a$ for each $n \in N \cup \{0\}$

(ii) For each $i \in I$ and $n \in N$:

$$p_i(x_{n+1} - x_n) \leqslant \sum_{r=1}^{k} q(i, r) p_{fr(i)}(x_n - x_{n-1})$$

Now, for any $i \in I$ and $n \in N$ it follows by (ii) that:

$$P(i, n, x_{m+1}, x_m) = \max\{p_{a(i)}(x_{m+1} - x_m) : a \in V(n, k)\} \leqslant$$

$$\leqslant \max\left\{\sum_{r=1}^{k} q(a(i), r) p_{f_r \circ a(i)}(x_m - x_{m-1}) : a \in V(n, k)\right\} \leqslant$$

$$\leqslant k Q(i, n) P(i, n+1, x_m, x_{m-1})$$

Consequently for any $i \in I$ we have:

$$p_i(x_{n+1} - x_n) = P(i, 0, x_{n+1}, x_n) \leqslant k^n \prod_{r=1}^{n} Q(i, r-1) P(i, n, x_1, x_0)$$

and because $R < \frac{1}{k}$ the series $\sum_{r=1}^{\infty} x_r - x_{r-1}$ is convergent. So there exists $x^* = \lim_{n \to \infty} x_n \in M$ and it remains to prove that $x^* \in Tx^*$. In order to prove this, we apply the condition 1. where $u = x_n$, $v = x^*$, $x = x_{n+1} \in Tx_n$. Then there exists $b_n \in Tx^*$, for every $n \in N$, such that for every $i \in I$ the following inequality holds:

$$p_i(x_{n+1} - b_n) \leqslant \sum_{r=1}^{k} q(i, r) p_{fr(i)}(x - x^*) \tag{1}$$

So from the relation $\lim_{n \to \infty} x_n = x^*$ and (1) we conclude that $\lim_{n \to \infty} p_i(b_n - x_{n+1}) = 0$ for every $i \in I$ which means that $\lim_{n \to \infty} b_n = \lim_{n \to \infty} x_n = x^*$. For every $n \in N$, $b_n \in Tx^*$ and so $x^* \in \overline{Tx^*}$. Since $Tx^* \in Cl(M)$ we have $Tx^* = \overline{Tx^*}$ which completes the proof.

Remark: It is easy to see that for every $i \in I$ we have:

$$p_i(x^* - x_0) \leqslant S(i, x_1, x_0) \text{ where:} \tag{2}$$

$$S(i, x_1, x_0) = P(i, 0, x_1, x_0) + \sum_{n=2}^{\infty} k^{n-1} P(i, n-1, x_1, x_0)$$

$$x \prod_{r=0}^{n-2} Q(i, r)$$

The inequality (2) will be used in the proof of Theorem 3.

Theorem 2. *Suppose that E, T, M, $f_r (=1, 2, \dots, k)$ and q are as in Theorem 1 and that the condition 1. in Theorem 1 is satisfied. Further, suppose that for every $i \in I$ and $n \in \cup \{0\}$ there exists $c_n(i) \geq 0$ and $g(i) \in I$ so that:*

$$p_{a(i)}(x) \leq c_n(i)\, p_{g(i)}(x) \qquad \textit{for every } x \in E$$

and every $a \in V(n, k)$ and:

$$\sup_{i \in I} \overline{\lim_{n \in N}} \sqrt[n]{c_n(i) \prod_{r=0}^{n-1} Q(i, r)} = \frac{1}{k}$$

If M is convex and sequentially compact there exists at least one element $x \in M$ such that $x \in Tx$.

Proof: For every $n \in N$ we shall define the mapping $T_n : M \to Cl(M)$ in the following way:

$$T_n x = \lambda_n Tx + (1-\lambda_n) x_0, \qquad \text{for every } x \in M$$

where x_0 is an arbitrary element from M and $\{\lambda_n\}_{n \in N}$ is a sequence of real numbers from $(0, 1)$ such that $\lim_{n \to \infty} \lambda_n = 1$. Since M is convex, we have $T_n x \in M$ for every $n \in N$ and every $x \in M$. Also, $T_n x \in Cl(M)$ because $Tx \in Cl(M)$ and E is a topological vector space. We shall show that the mapping T_n, for every $n \in N$, satisfies all the conditions of Theorem 1. Let $u, v \in M$ and $x \in T_n u = \lambda_n Tu + (1-\lambda_n) x_0$. Then $x' = \frac{x-(1-\lambda_n)x_0}{\lambda_n} \in Tu$. Since the mapping T satisfies the condition 1. of Theorem 1 there exists element $y \in Ty$ so that for every $i \in I$ we have:

$$p_i(x'-y) \leq \sum_{r=1}^{k} q(i, r)\, p_{f_r(i)}(u-v)$$

namely that:

$$p_i\left(\frac{x-(1-\lambda_n)x_0}{\lambda_n} - y\right) \leq \sum_{r=1}^{k} q(i, r)\, p_{f_r(i)}(u-v)$$

From this it follows that:

$$p_i(x-((1-\lambda_n)x_0+\lambda_n y)) \leq \sum_{r=1}^{k} \lambda_n q(i, r)\, p_{f_r(i)}(u-v)$$

where $(1-\lambda_n)x_0+\lambda_n y\in T_n v$. Further we have for every $n\in N$:

$$R_n=\sup_{i\in I}\overline{\lim_{m\in N}}\sqrt[m]{P(i,\ m,\ x_1,\ x_0)\prod_{r=0}^{m-1}Q_n(i,\ r)}\leqslant$$

$$\leqslant\sup_{i\in I}\overline{\lim_{m\in N}}\sqrt[m]{c_m(i)\,\lambda_n^m\prod_{r=0}^{m-1}Q(i,\ r)}\leqslant\frac{1}{k}$$

where $Q_n(i,\ m)=\max\{\lambda_n\ q(a(i),\ r):r\in A,\ a\in V(m,\ k)\}$ for every $m\in N$ and $Q_n(i,\ 0)=\lambda_n\max\{q(i,\ r):r\in A\}$, $x_1\in T_n x_0$. So for every $n\in N$ there exists $x_n\in M$ such that:

$$x_n\in T_n x_n=\lambda_n\,Tx_n+(1-\lambda_n)\,x_0.$$

Since the set M is a sequentially compact subset of E there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of the sequence $\{x_n\}_{n\in N}$ such that $\lim_{k\to\infty}x_{n_k}=x^*$. It remains to prove that $x^*\in Tx^*$. From $x_n\in T_n x_n=\lambda_n Tx_n+(1-\lambda_n)x_0$ it follows that:

$$\frac{x_n-(1-\lambda_n)\,x_0}{\lambda_n}\in Tx_n.$$

Now, if we apply the condition 1 of Theorem 1 where:

$$u=x_n,\ \ v=x^*,\ \ x=\frac{x_n-(1-\lambda_n)\,x_0}{\lambda_n}$$

it follows that there exists element $b_n\in Tx^*$, for every $n\in N$ such that for every $i\in I$ the following inepuality holds:

$$p_i\left(\frac{x_n-(1-\lambda)\,x_0}{\lambda_n}-b_n\right)\leqslant\sum_{r=1}^{k}q(i,\ r)\,p_{f_k(i)}(x_n-x^*).$$

Since $\lim_{k\to\infty}x_{nk}=x^*$ we have that $\lim_{k\to\infty}b_{nk}=\lim_{k\to\infty}\frac{x_{nk}-(1-\lambda_{nk})\,x_0}{\lambda_{nk}}=\lim_{k\to\infty}x_{nk}=x^*$. So we have $x^*\in Tx^*=\overline{Tx^*}$.

If we have a family of multifunctions $\{F_\lambda\}_{\lambda\in\Lambda}$, where Λ is a topological space, such that for every $\lambda\in\Lambda$ there exists a fixed point x_λ of the mapping F_λ i.e. $x_\lambda\in F_\lambda x_\lambda$ it is of interest to investigate "the continuous dependence" of the fixed point x_λ on parameter $\lambda\in\Lambda$. Here "the continuous dependence" of the fixed point x_λ on parameter $\lambda\in\Lambda$ means that for every $i\in I$, every $\varepsilon>0$ and every $\lambda_0\in\Lambda$ there exists a neighbourhood $V(\lambda_0,\ i,\ \varepsilon)$ of λ_0 such that for every $\lambda\in V(\lambda_0,\ i,\ \varepsilon)$ there exist $x_\lambda\in F_\lambda\,x_\lambda$ and $x_{\lambda_0}\in F_{\lambda_0}\,x_{\lambda_0}$ such that

$$p_i(x_\lambda-x_{\lambda_0})<\varepsilon$$

Definition. *Let Λ be a topological space, E be a locally convex space, M be a subset of E and for every $\lambda\in\Lambda$, $F_\lambda:M\to 2^E$. We write $F_\lambda\overset{w}{\to}F_\lambda$, when $\lambda\to\lambda_0$ iff the following condition is satisfied:*

For every $x \in M$, *every* $i \in I$, *every* $\varepsilon > 0$ *and every* $z \in F_{\lambda_0} x$ *there exists a neighbourhood* V *of* λ_0 *such that:*

$$F_\lambda x \cap W \neq \varnothing \qquad \text{for every } \lambda \in V$$

where $W = \{u : p_i(u-z) < \varepsilon\}$.

For singlevalued mappings $F_\lambda \xrightarrow{w} F_{\lambda_0}$ when $\lambda \to \lambda_0$ if $\lim_{\lambda \to \lambda_0} F_\lambda x = F_{\lambda_0} x$ for every $x \in M$.

If $f_{r,\lambda} : I \to I$ for every $r \in A$ and $\lambda \in \Lambda$ we shall denote $f_{J_1,\lambda} \circ f_{J_2,\lambda} \circ \cdots f_{J_n,\lambda}(i)$ by $a_\lambda(i)$ $(a = (J_1, J_2, \ldots, J_n \in A)$ for every $\lambda \in \Lambda$. Further if $q_\lambda : IxA \to R^+$ we shall use the notation:

$$Q_\lambda(i, n) = \max\{q_\lambda(a_\lambda(i), r) : a \in V(n, k),\ r \in A\}$$

$$Q_\lambda(i, 0) = \max\{q_\lambda(i, r) : r \in A\}$$

$$\overline{Q}(i, n) = \max_{\lambda \in \Lambda} Q_\lambda(i, n).$$

Now, we shall give a sufficient condition for continuous dependence of the fixed point x_λ of the mapping F_λ on parameter $\lambda \in \Lambda$.

Theorem 3. *Suppose that* E *and* M *are as in Theorem* 1 Λ *is a topological space,* $F_\lambda : M \to C1(M)$ *for every* $\lambda \in \Lambda$ *and* $F_\lambda \xrightarrow{w} F_{\lambda_0}$ *if* $\lambda \to \lambda_0$ *for every* $\lambda_0 \in \Lambda$. *Further suppose that for every* $\lambda \in \Lambda$ *the mapping* F_λ *satisfies the condition* 1. *of Theorem* 1 *and that the following two conditions are satisfied:*

(i) $p_{a_\lambda(i)}(x) \leqslant c_n(i)\, p_{g(i)}(x)$ *for every* $i \in I$, $n \in N$, *every* $x \in E$, *every* $\lambda \in \Lambda$ *and every* $a \in V(n, k)$

(ii) $\sup_{i \in I} \overline{\lim_{n \in N}} \sqrt[n]{c_n(i) \prod_{r=0}^{n-1} \overline{Q}(i, r)} < \frac{1}{k}$.

Then for every $i \in I$, *every* $\varepsilon > 0$ *and every* $\lambda_0 \in \Lambda$ *there exists a neighbourhood* $V(\lambda_0, i, \varepsilon)$ *of* λ_0 *so that for every* $\lambda \in V(\lambda_0, i, \varepsilon)$ *there exists a fixed point* x_λ *of the mapping* F_λ *and a fixed point* x_{λ_0} *of the mapping* F_{λ_0} *such that:*

$$p_i(x_\lambda - x_{\lambda_0}) < \varepsilon$$

Proof: From the conditions (i) and (ii) it follows that for every $\lambda \in \Lambda$ there exists a fixed point x_λ of the mapping F_λ and $x_\lambda = \lim_{n \to \infty} x_{n,\lambda}$, $x_{n,\lambda} \in F_\lambda x_{n-1,\lambda}$ where $x_{0,\lambda}$ is an arbitrary element from M and $x_{1,\lambda}$ is an arbitrary element from $F_\lambda x_{0,\lambda}$. Let $x_{0,\lambda} = x_{\lambda_0} \in F_{\lambda_0} x_{\lambda_0}$ for every $\lambda \in \Lambda$. Then we have from inequality (2):

$$p_i(x_\lambda - x_{\lambda_0}) \leqslant p_{g(i)}(x_{1,\lambda} - x_{\lambda_0}) \left(c_0(i) + \sum_{s=1}^{\infty} c_s(i)\, k^s \prod_{t=0}^{s-1} \overline{Q}(i, t) \right)$$

where $x_{1,\lambda} \in F_\lambda x_{\lambda_0}$. Let us denote $c_0(i) + \sum_{s=1}^{\infty} c_s(i) k^s \prod_{t=0}^{s-1} \overline{Q}(i, t)$ by $M(i)$. Now we shall use the Definition where:

$$x = x_{\lambda_0} \text{ and } z = x_{\lambda_0} \in F_{\lambda_0} x_{\lambda_0}$$

Then there exists a neighbourhood V of λ_0 such that:

$$F_\lambda x_{\lambda_0} \cap W \neq \varnothing \qquad \text{for every } \lambda \in \Lambda$$

where $W = \left\{ u : p_i(u - x_{\lambda_0}) < \frac{\varepsilon}{M(i)} \right\}$. This means that there exists $x_{1,\lambda} \in F_\lambda x_{\lambda_0}$ such that:

$$p_i(x_{1,\lambda} - x_{\lambda_0} < \frac{\varepsilon}{M(i)}$$

and so $p_i(x_\lambda - x_{\lambda_0}) < \varepsilon$ for every $\lambda \in V$, every $i \in I$ and every $\varepsilon > 0$.

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