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CONTRIBUTION TO THE PROBLEM OF THE SPECTRA OF COMPOUND GRAPHS

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The investigation of the spectral properties of graphs formed by a given operation from simpler graphs is a problem of some interest in graph theory. [1—4, 8] In the present work the characteristic and acyclic polynomials will be determined for certain types of compound graphs.

The characteristic polynomial [2] of a graph G will be denoted by $\Phi(G)=\Phi(G, \lambda)$. The acyclic polynomial $\alpha(G)=\alpha(G, \lambda)$ of a graph G is defined in [6]. Our notation and terminology completely follows that of [6] and will not be introduced here once again.

Let P_k be a path with k vertices $v_1, v_2, \dots, v_k$, such that v_j and v_{j+1} are adjacent $(j=1, 2, \dots, k-1)$. Let further the edge connecting the vertices v_j and v_{j+1} be denoted by e_j.

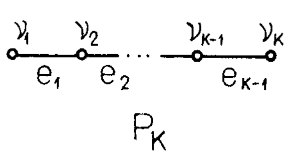

Figure 1

An edge e in *a* graph G is called *a* bridge if $G-e$ has more components than G.

Definition 1. A graph Q_k belongs to the class $\mathbf{Q}_k$ if, and only if it contains P_k as a subgraph and if its edges e_j are bridges for all $j=1. 2, \dots, k-1$.

Consequently, every graph with at least one vertex belongs to $\mathbf{Q}_1$; every graph with a bridge belongs to $\mathbf{Q}_2$ etc. In the general case, a graph from $\mathbf{Q}_k$ has the following structure.

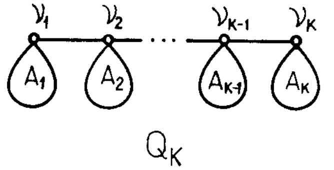

Figure 2

$A_1, A_2, \ldots, A_k$ symbolize arbitary, mutually disconnected subgraphs. Hence Q_k can be understood as being composed of the fragments $A_1, A_2, \ldots, A_k$. The path P_k is now a special case of a graph Q_k. The subgraph obtained by deletion of the vertex v_j from A_j will be denoted by B_j.

The fact that two graphs G_1 and G_2 are isomorphic is written as $G_1 = G_2$.

Definition 2. A graph R_k belongs to the class $\mathbf{R}_k$ if, and only if $R_k \in \mathbf{Q}_k$ and $A_2 = A_3 = \cdots = A_{k-1}$ and $B_2 = B_3 = \cdots = B_{k-1}$. A graph $\mathbf{R}_k^*$ belongs to the class $\mathbf{R}_k^*$ if, and only if $R_k^* \in \mathbf{R}_k$ and $A_k = A_{k-1}$ and $B_k = B_{k-1}$. A graph S_k belongs to the class $\mathbf{S}_k$ if, and only if $S_k \in \mathbf{R}_k^*$ and $A_1 = A_2$ and $B_1 = B_2$.

Of course, it is $\mathbf{S}_k \subseteq \mathbf{R}_k^* \subseteq \mathbf{R}_k \subseteq \mathbf{Q}_k$ and the graphs R_k, R_k^* and S_k are special cases of graphs Q_k. In the following, the mutually isomorphic subgraphs A_j and B_j, contained in the graphs R_k, R_k^* and S_k will be denoted simply by A and B.

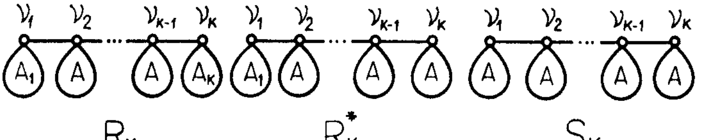

Figure 3

The characteristic polynomials of graphs Q_k, R_k and S_k

Since e_{k-1} is a bridge, the characteristic polynomial of Q_k fulfils the equality [5]

(1) $$\Phi(Q_k) = \Phi(Q_k - e_{k-1}) - \Phi(Q_k - (e_{k-1}))$$

i.e.

(2) $$\Phi(Q_k) = \Phi(A_k)\,\Phi(Q_{k-1}) - \Phi(B_{k-1})\,\Phi(B_k)\,\Phi(Q_{k-2})$$

This latter recursion relation can be written in a matrix form as

$$\begin{bmatrix} \Phi(Q_k) \\ \Phi(B_k)\,\Phi(Q_{k-1}) \end{bmatrix} = \begin{bmatrix} \Phi(A_k) & -\Phi(B_k) \\ \Phi(B_k) & 0 \end{bmatrix} \begin{bmatrix} \Phi(Q_{k-1}) \\ \Phi(B_{k-1})\,\Phi(Q_{k-2}) \end{bmatrix} \tag{3}$$

A repeated application of eq. (3) leads immediately to the following result.

Theorem 1.

$$\begin{bmatrix} \Phi(Q_k) \\ \Phi(B_k)\,\Phi(Q_{k-1}) \end{bmatrix} = T_k T_{k-1} \cdots T_2 T_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4}$$

where

$$T_j = \begin{bmatrix} \Phi(A_j) & -\Phi(B_j) \\ \Phi(B_j) & 0 \end{bmatrix} \quad j = 1, 2, \ldots, k$$

Corollary 1.1. *The characteristic polynomial of Q_k is completely determined by the characteristic polynomials of A_j and B_j $(j=1, \ldots, k)$. This implies the existence of numerous pairs (triplets etc.) of nonisomorphic isospectrai graphs in the class $\mathbf{Q}_k$ $(k \geqslant 2)$.*

A simple example is the pair G' and G''. Note that according to Definition 2, both G' and G'' belong to the class $\mathbf{S}_2$.

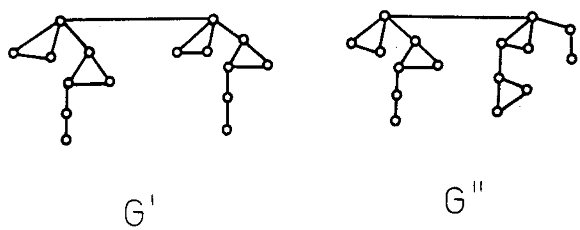

Figure 4

Corollary 1.2.

$$\begin{bmatrix} \Phi(R_k) \\ \Phi(B_k)\,\Phi(R^*_{k-1}) \end{bmatrix} = T_k (T)^{k-2}\, T_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \Phi(S_k) \\ \Phi(B)\,\Phi(S_{k-1}) \end{bmatrix} = (T)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where

$$T = \begin{bmatrix} \Phi(A) & -\Phi(B) \\ \Phi(B) & 0 \end{bmatrix}$$

Corollary 1.3.

$$\begin{bmatrix} \Phi(P_k) \\ \Phi(P_{k-1}) \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A result of the form (4) was recently obtained by Kaulgud and Chitgopkar. [7]

The above defined and investigated graphs are special cases of so called "rooted products". [4] In particular, Q_k is the rooted product of P_k with the sequences of graphs $A_1, A_2, \ldots, A_k$. Schwenk [8] and later Godsil and McKay [4] found general expressions for the characteristic polynomials of rooted products. Our results, on the other hand, emphasize certain special properties of the rooted products of a path, which were not considered previously.

In the special case of graphs S_k, eq. (2) is written as

$$\Phi(S_k) = \Phi(A)\,\Phi(S_{k-1}) - \Phi(B)^2\,\Phi(S_{k-2}) \tag{5}$$

Theorem 2. *A recursion relation of the form* (5) *holds for all* $R_k \in \mathbf{R}_k$, *namely*

$$\Phi(R_k) = \Phi(A)\,\Phi(R_{k-1}) - \Phi(B)^2\,\Phi(R_{k-2}) \tag{6}$$

Proof. According to (2) and Definition 2 we have

$$\begin{aligned}\Phi(R_k) &= \Phi(A_k)\,\Phi(R^*_{k-1}) - \Phi(B)\,\Phi(B_k)\,\Phi(R^*_{k-2}) = \\ &= \Phi(A_k)\,[\Phi(A)\,\Phi(R^*_{k-2}) - \Phi(B)^2\,\Phi(R^*_{k-3})] - \\ &- \Phi(B)\,\Phi(B_k)\,[\Phi(A)\,\Phi(R^*_{k-3}) - \Phi(B)^2\,\Phi(R^*_{k-4})] = \\ &= \Phi(A)\,[\Phi(A_k)\,\Phi(R^*_{k-2}) - \Phi(B)\,\Phi(B_k)\,\Phi(R^*_{k-3})] - \\ &- \Phi(B)^2\,[\Phi(A_k)\,\Phi(R^*_{k-3}) - \Phi(B)\,\Phi(B_k)\,\Phi(R^*_{k-4})] = \\ &= \Phi(A)\,\Phi(R_{k-1}) - \Phi(B)^2\,\Phi(R_{k-2}).\end{aligned}$$

From eq. (6) it follows

$$\Phi(R_k) = C_1\,(t_1)^k + C_2\,(t_2)^k \tag{7}$$

where t_1 and t_2 are the solutions of the equation $t^2 = \Phi(A)\,t - \Phi(B)^2$. By setting $\Phi(A) = 2\,\Phi(B)\cos\theta$, one gets $t_{1,2} = \Phi(B)\exp(\pm i\theta)$, which substituted back into (7) gives the general expressions for $\Phi(R_k)$ and $\Phi(S_k)$.

Corollary 2.1. *For all* $R_k \in \mathbf{R}_k$ *it is*

$$\Phi(R_k) = \Phi(R_2)\,\Phi(B)^{k-2}\,\frac{\sin(k-1)\theta}{\sin\theta} - \Phi(R_1)\,\Phi(B)^{k-1}\,\frac{\sin(k-2)\theta}{\sin\theta} \tag{8}$$

while for all $S_k \in \mathbf{S}_k$ *it is*

$$\Phi(S_k) = \Phi(B)^k\,\frac{\sin(k+1)\theta}{\sin\theta} \tag{9}$$

Note that (9) is obtained from (8) because of $\Phi(S_1) = \Phi(A)$, $\Phi(S_2) = \Phi(A)^2 - \Phi(B)^2$ and $\Phi(A) = 2\,\Phi(B)\cos\theta$. Combination of (8) and (9) yields finally

$$\Phi(R_k) = \Phi(R_2)\,\Phi(S_{k-2}) = \Phi(R_1)\,\Phi(B)^2\,\Phi(S_{k-3}) \tag{10}$$

Corollary 2.2. *If the subgraph A possesses exactly one vertex, then we have* $\Phi(A) \equiv \lambda$, $\Phi(B) \equiv 1$ *and* (6) *becomes* $\Phi(R_k) = \lambda\Phi(R_{k-1}) - \Phi(R_{k-2})$.

Equation (9) gives the well known relation [2] $\Phi(P_k, 2\cos\theta) = \sin(k+1)\theta / \sin\theta$, while eq. (10) reduces to $\Phi(R_k) = \Phi(R_2)\,\Phi(P_{k-2}) - \Phi(R_1)\,\Phi(P_{k-3})$.

These special cases of Theorem 2 were proved in [5].

The following formulas can be checked by straightforward application of Theorem 2.

Corollary 2.3. *Let* $r_k = \Phi(R_k)^2 - \Phi(R_{k-1})\,\Phi(R_{k+1})$ *and* $s_k = \Phi(S_k)^2 - \Phi(S_{k-1})\,\Phi(S_{k+1})$. *Then*

$$r_k = \Phi(B)^2\, r_{k-1}; \qquad r_k = \Phi(B)^{2(k-2)}\, r_2$$

$$s_k = \Phi(B)^2\, s_{k-1}; \qquad s_k = \Phi(B)^{2k}$$

The latter equality is a consequence of $s_2 = \Phi(B)^4$.

Because of the identity $\dfrac{\sin(k+1)\theta}{\sin\theta} = 2^k \prod_{j=1}^{k}\left(\cos\theta - \cos\dfrac{j\pi}{k+1}\right)$, eq. (9) is further transformed into

$$\Phi(S_k) = \prod_{j=1}^{k}\left[\Phi(A) - 2\,\Phi(B)\cos\frac{j\pi}{k+1}\right] \tag{11}$$

Formula (11) was obtained by Schwenk [8] and recently also by Godsil and McKay [4], but using a different way of reasoning.

The acyclic polynomials of graphs Q_k, R_k and S_k

The acyclic polynomial fulfils the recurrence formula [6]

$$\alpha(G) = \alpha(G-e) - \alpha(G-(e))$$

where e is an arbitrary edge of an arbitrary graph G. Therefore also $\alpha(Q_k)$ obeys the relation

$$\alpha(Q_k) = \alpha(Q_k - e_{k-1}) - \alpha(Q_k - (e_{k-1})) \tag{12}$$

which is fully analogous to eq. (1). Since all the statements obtained in the previous section are consequences of eq. (1), a completely equivalent reasoning based on eq. (12) leads to the same results also for the acyclic polynomial.

Hence, all equations, theorems and corollaries of the preceeding section remain valid if the term "characteristic polynomial, Φ" is substituted by "acyclic polynomial, α". In particular, identity (11) is now reformulated as follows.

Theorem 3a. *The set of all zeros of the acyclic polynomial of a graph* $S_k \in \mathbf{S}_k$ *is composed of the solutions of the equations* $\alpha(A) = 2\,\alpha(B)\cos\frac{j\pi}{k+1}$ *for* $j = 1, 2, \ldots, k$.

The characteristic and acyclic polynomials of graphs U_k

Definition 3. A graph U_k belongs to the class $\mathbf{U}_k$ if, and only if it is obtained from a graph $S_k \in \mathbf{S}_k$ by introducing a new edge between the vertices v_1 and v_k.

Note that the cycle C_k with k vertices is a special case of the graphs U_k. The general form of a graph $U_k \in \mathbf{U}_k$ is the following.

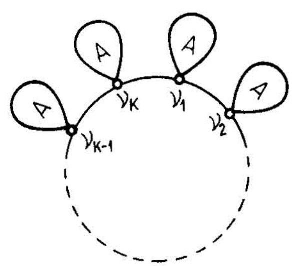

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Figure 5

The characteristic and acyclic polynomials of the graph U_k fulfil the relations [2, 6]

$$\Phi(U_k) = \Phi(S_k) - \Phi(B)^2\,\Phi(S_{k-2}) - 2\,\Phi(B)^k \tag{13}$$

$$\alpha(U_k) = \alpha(S_k) - \alpha(B)^2\,\alpha(S_{k-2}) \tag{14}$$

Therefrom one can determine the following recursion formulas.

$$\Phi(U_k) + 2\,\Phi(B)^k = \Phi(A)\,[\Phi(U_{k-1}) + 2\,\Phi(B)^{k-1}] - \Phi(B)^2\,[\Phi(U_{k-2}) + 2\,\Phi(B)^{k-2}]$$

$$\alpha(U_k) = \alpha(A)\,\alpha(U_{k-1}) - \alpha(B)^2\,\alpha(U_{k-2})$$

These have the same structure (and therefore also the same consequences) as eqs. (5) and (6). The general form of $\Phi(U_k)$ and $\alpha(U_k)$ can be deduced by combining eqs. (13) and (14) with (9).

$$(15)\qquad \Phi(U_k)=2\ \Phi(B)^k(\cos k\theta-1)=\prod_{j=1}^{k}\left[\Phi(A)-2\ \Phi(B)\cos\frac{2j\pi}{k}\right]$$

$$\alpha(U_k)=2\ \alpha(B)^k\cos k\theta=\prod_{j=1}^{k}\left[\alpha(A)-2\ \alpha(B)\cos\frac{(2j+1)\pi}{k}\right]$$

Equation (15) was also derived by Schwenk by means of different methods.

Theorem 3b. *The set of all zeros of the acyclic polynomial of a graph* $U_k\in\mathbf{U}_k$ *is composed of the solutions of the equations* $\alpha(A)=2\ \alpha(B)\cos\frac{(2j+1)\pi}{k}$ *for* $j=1, 2, \ldots, k$.

We conclude this work with two propositions which reflect certain algebraic properties of the characteristic polynomials of matrices and graphs. Both are simple consequences of eqs. (11) and (15).

Let M be a symmetric square matrix with real (but arbitrary) elements. Let M_j be the submatrix obtained by deletion of the j-th row and the j-th column from M. Let $\Phi(M)$ and $\Phi(M_j)$ be the characteristic polynomials of M and M_j.

Proposition 1. For a and b being arbitrary integers ($b\neq 0$), all solutions of the equation $\Phi(M)=2\ \Phi(M_j)\cos\frac{a\pi}{b}$ are real.

Let G be an arbitrary graph and $G-v$ the subgraph obtained by deletion of the vertex v from G.

Proposition 2. For a and b being arbitrary integers ($b\neq 0$), all solutions of the equation $\Phi(G)=2\ \Phi(G-v)\cos\frac{a\pi}{b}$ belong to the spectrum of some graph.

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