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ON COMON FIXED POINTS IN UNIFORM SPACES

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Recently Acharya [1], [2] proved some fixed point theorems of mappings of a uniform space into itself. In this paper we define a new condition of common contractivity for a pair of mappings of a metrizable space into itself and then prove some theorems about common fixed points of family of contractive maps on a uniform space which are generalizations of results in [1], [6], [10].

First of all, we give the following result for metrizable spaces.

Theorem 1. *Let X be a metrizable uniform space and F and T be a pair of selfmappings of X. If (X, d), for some metric d, is complete and the mappings F and T satisfy the condition*

$$d(Fx, Ty) \leqslant q \cdot \max \left\{ d(x\, y), \frac{1}{2} d(x, Fx), \frac{1}{2} d(y, Ty), d(x, Ty), d(y, Fx) \right\}$$

for some $q<1$ and all $x, y \in X$, then F and T have a unique common fixed point.

Proof. Let x be an arbitrary point in X. We shall show that the sequence $\{T^n x\}_{n=0}^{\infty}$ is bounded. For an arbitrary integer n let $k=k(n)$ be such that

$$d(Fx, T^k x) = \max \{d(Fx, T^i x) : i=1, 2, \ldots, n\}.$$

Then

$$d(Fx, T^n x) \leqslant d(Fx, T^k x)$$

$$\leqslant q \max \left\{ d(x, T^{k-1} x), \frac{1}{2} d(x, Fx), \frac{1}{2} d(T^{k-1} x, T^k x), d(x, T^k x), d(Fx, T^{k-1} x) \right\}$$

$$\leqslant q \max \left\{ d(x, Fx) + d(Fx, T^{k-1} x), \frac{1}{2} d(x, Fx), \frac{1}{2} [d(T^{k-1} x, Fx) + d(Fx, T^k x)], \right.$$

$$\left. d(x. Fx) + d(Fx, T^k x), d(Fx, T^{k-1} x) \right\}$$

$$\leqslant q \max \left\{d(x, Fx)+d(Fx, T^k x), \frac{1}{2}[d(T^k x, Fx)+d(Fx, T^k x)], d(Fx, T^k x)\right\}=$$

$$=q[d(x, Fx)+d(Fx, T^k x)]$$

and hence

$$d(Fx, T^k x)\leqslant \frac{q}{1-q} d(x, Fx).$$

Therefore, for any integer n we have

$$d(Fx, T^n x)\leqslant d(Fx, T^{k(n)} x)\leqslant \frac{q}{1-q} d(x, Fx).$$

Hence we conclude that the sequence $\{T^n x\}_{n=0}^{\infty}$ is bounded.

Similarly, the sequence $\{F^n x\}_{n=0}^{\infty}$ is bounded and so

$$\delta_n := \sup \{d(F^i x, T^j x): i, j \geqslant n\} < \infty.$$

For $i, j \geqslant n$ we have

$$d(F^i x, T^j x)\leqslant q\cdot \max\left\{d(F^{i-1} x, T^{j-1} x), \frac{1}{2} d(F^{i-1} x, F^i x), \frac{1}{2} d(T^{j-1} x, T^j x),\right.$$

$$\left. d(F^{i-1} x, T^j x), d(F^i x, T^{j-1} x\right\}$$

$$\leqslant q\cdot \max \left\{d(F^{i-1} x, T^{j-1}), \frac{1}{2}[d(F^{i-1} x, T^{j-1} x)+d(T^{j-1} x, F^i x],\right.$$

$$\left.\frac{1}{2}[d(T^{j-1} x, F^{i-1} x)+d(F^{i-1} x, T^j x)], d(F^{i-1} x, T^j x), d(F^i x, T^{j-1} x)\right\}$$

$$\leqslant q\cdot \max \left\{d(F^{i-1} x, T^{j-1} x), d(T^{j-1} x, F^i x), d(F^{i-1} x, T^j x)\right\}\leqslant q\cdot \delta_{n-1}$$

and hence

$$\delta_n \leqslant q\cdot \delta_{n-1}.$$

This implies that $\lim_{n\to\infty} \delta=0$. Since

$$d(T^n x, T^{n+p} x)\leqslant d(F^n x, T^n x)+d(F^n x, T^{n+p} x)\leqslant 2\,\delta_n,$$

it follows that $\{T^n x)_{n=0}^{\infty}$ is a Cauchy sequence in the complete metric space (X, d) and so has a limit u in X. Since

$$d(u, F^n x)\leqslant d(u, T^n x)+d(T^n x, F^n x)\leqslant d(u, T^n x)+\delta_n$$

it follows that $u=\lim_{n\to\infty} F^n x$, too.

Now we have

$$d(u, Tu) \leqslant d(u, F^{n+1}x) + d(FF^n x, Tu) \leqslant d(u, F^{n+1}x) +$$

$$q \cdot \max\left\{d(F^n x, u), \frac{1}{2} d(F_n x, F^{n+1}x), \frac{1}{2} d(u, Tu), d(F^n x, Tu), d(F^{n+1}x, u)\right\}$$

and on letting n tend to infinity we see that

$$d(u, Tu) \leqslant q \cdot d(u, Tu).$$

Since $q<1$, it follows that $d(u, Tu)=0$, i.e. u is a fixed point of T.

Similarly, u is also a fixed point of F.

Now suppose that v is an another common fixed point of F and T. Then

$$d(u, v) = d(Fu, Tv) \leqslant q \cdot \max\left\{d(u, v), \frac{1}{2} d(u, Fu), \frac{1}{2} d(v, Tv), d(u, Tv), d(v, Fu)\right\} =$$

$$= q \cdot d(u, v)$$

and hence $v=u$. Assume now that $w=Fw$. Then

$$d(w, Tw) = d(Fw, Tw) \leqslant q \cdot \max\left\{\frac{1}{2} d(w, Tw), d(w, Tw)\right\} = q \cdot d(w, Tw)$$

and since $q<1$, $d(w, Tw)=0$. Therefore the uniqueness of u follows. This completes the proof of the theorem 1.

Now we shall extend this result to uniform spaces which need not be metrisable.

Let $(X, \mathcal{U})$ be a uniform space. For any pseudometric p on X and any $r>0$, let $V(p, r)=\{(x, y): x, y \in X \text{ and } p(x, y)<r\}$. Let P be the family of pseudometrics on X generating the uniformity $\mathcal{U}$. Then the family $\mathcal{V}$ of all sets of the form $\bigcap_{i=1}^{n} V(p_i, r_i)$ (the integer n is not fixed), where $p_i \in P$, $r_i>0$, $i=1, 2, \ldots, n$, forms a base for the uniformity $\mathcal{U}$. For each $a>0$, the set $\bigcap_{i=1}^{n} V(p_i, ar_i)$ belongs to $\mathcal{V}$. This set is denoted by aV, where $V=\bigcap_{i=1}^{n} V(p_i, r_i)$. For other properties of the uniformity used in this paper the reader may consult [1] and [2].

Theorem 2. *Let X be a sequential complete Hausdorff uniform space and F and T be a pair of selfmappings of X. If for any $V_i \in \mathcal{V}$ $(i=1, 2, 3, 4, 5)$ and $x, y \in X$*

$$(x, y) \in V_1, \ (x, Fx) \in V_2, \ (y, Ty) \in V_3, \ (x, Ty) \in V_4, \ (y\ Fx) \in V_5$$

implies

$$(1) \qquad (Fx, Ty) \in aV_1 \circ bV_2 \circ cV_3 \circ eV_4 \circ fV_5,$$

for some nonnegative functions $a=a(x, y)$, $b=b(x, y)$, $c=c(x, y)$, $e=e(x, y)$ *and* $f=f(x, y)$ *satisfying*

$$a(x, y)+2b(x, y)+2c(x, y)+e(x, y)+f(x, y) \leqslant q<1,$$

then F and T have a unique common fixed point.

Proof. Let x, y in X and V in $\mathcal{U}$ be arbitrary. Choose p to be the Minkowski pseudometric corresponding to V. Put $p(x, y)=t_1$, $p(x, Fx)=t_2$, $p(y, Ty)=t_3$, $p(x, Ty)=t_4$, $p(y, Fx)=t_5$ and let $\varepsilon>0$. Then

$$(x, y)\in(t_1+\varepsilon)V_1,\quad (x,Fx)\in(t_2+\varepsilon)V_2,\quad (y, Ty)\in(t_3+\varepsilon)V_3,\quad (x, Ty)\in(t_4+\varepsilon)V_4,$$
$$(y, Fx)\in(t_5+\varepsilon)V_5,$$

which imply that

$$(Fx, Ty)\in a(t_1+\varepsilon)V_1\circ b(t_2+\varepsilon)V_2\circ c(t_3+\varepsilon)V_3\circ e(t_4+\varepsilon)V_4\circ f(t_5+\varepsilon)V_5$$

and hence

$$p(Fx, Ty)<a(t_1+\varepsilon)+b(t_2+\varepsilon)+c(t_3+\varepsilon)+e(t_4+\varepsilon)+f(t_5+\varepsilon).$$

Since ε is arbitrary,

$$p(Fx, Ty)\leqslant ap(x, y)+2b\frac{1}{2}p(x, Fx)+2c\frac{1}{2}p(y, Ty)+ep(x, Ty)+fp(y, Fx)$$

$$\leqslant(a+2b+2c+e+f)\max\left\{p(x, y), \frac{1}{2}p(x, Fx), \frac{1}{2}p(y, Ty), p(x, Ty), p(y, Fx)\right\}$$

$$\leqslant q\max\left\{p(x, y), \frac{1}{2}p(x, Fx), \frac{1}{2}p(y, Ty), p(x, Ty), p(y, Fx)\right\}.$$

Using the same argument as in the theorem 1 we obtain that $p(u, Tu)=p(u, Fu)=0$ for some u in X. Therefore (u, Tu), $(u, Fu)\in V$ for every V in $\mathcal{U}$. Hence $Tu=u=Fu$. The uniqueness of u follows by the same argument as in the theorem 1.

Corollary 1. *Let F and T be mappings of a sequential complete Hausdorf uniform space X into itself. If there exist positive integers i and j such that* F^i *and* T^j *satisfy* (1), *then F and T have a unique common fixed point.*

Similarly we can extend results given in [6] and [8] from metric spaces to uniform spaces. Here will be stated the extension of Theorem 2 of [6] only.

Theorem 3. *Let* $\mathcal{F}$ *be a family of selfmappings of a sequential complete Hausdorff uniform space X. If there exists some F in* $\mathcal{F}$ *such that for each* $T\in\mathcal{F}$ *there are positive integers* $i=i(T)$ *and* $j=j(T)$ *such that* $(x, y)\in V_1$, $(x, F^i x)\in V_2$, $(y, T^j y)\in V_3$, $(x, T^j y)\in V_4$, $(y, F^i x)\in V_5$ *imply*

$$(F^i x, Ty^j)\in aV_1\circ bV_2\circ cV_3\circ eV_4\circ eV_5$$

for all $x, y \in X$ *and* $V_i \in \mathcal{V}$ $(i=1, 2, 3, 4, 5)$, *where* a, b, c *and* e *are non-negative functions of* x *and* y *satisfying*

$$a(x, y)+b(x, y)+c(x, y)+2e(x, y) \quad \leqslant q<1,$$

then every $T \in \mathcal{F}$ *has a unique fixed point in* X; *at the same time, the same point is a common fixed point for* $\mathcal{F}$.

We conclude this paper with an open question.

If (F, T) is a pair of maps of a complete metric space M into itself satisfying

$$d(Fx, Ty) \leqslant q \cdot \max\{d(x, y), d(x, Fx), d(y, Ty), d(x, Ty), d(y, Fx)\}$$

for some $q<1$, do F and T have a common fixed point?

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