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ON PROBABILITY FUNCTION ON PSEUDO-BOOLEAN ALGEBRA

Jelena Bulatović

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Let $\mathcal{A}=(A, \cup, \cap, \rightarrow, -)$ be any pseudo-Boolean algebra, in which the operations $\cup, \cap, \rightarrow$ and $-$ are defined as in [1]; denote by $\varnothing$ and Ω, the minimal and maximal elements in A, respectively, and by $\leqslant$ the ordering relation in A (note that $a \leqslant b$ has the same meaning as $b \geqslant a$). The pseudo--difference of elements a and b from A, which will be denoted by $a \stackrel{*}{-} b$, we shall define by the equality

$$a \stackrel{*}{-} b = a \cap -b$$

(we always assume that the sign „$-$" binds more strongly than all the other signs); the pseudo-difference exists for all $a, b \in A$. It is easy to see that the equalities

$$\Omega \stackrel{*}{-} a = a \rightarrow \varnothing = -a$$

hold for each $a \in A$.

L e m m a 1. *In the pseudo-Boolean algebra the following statements are valid:*

(1′) $(a \cup b) \stackrel{*}{-} c = (a \stackrel{*}{-} c) \cup (b \stackrel{*}{-} c)$,

(1″) $(a \cap b) \stackrel{*}{-} c = (a \stackrel{*}{-} c) \cap b = (b \stackrel{*}{-} c) \cap a$;

(2′) $(c \stackrel{*}{-} a) \stackrel{*}{-} b = c \stackrel{*}{-} (a \cup b)$,

(2″) $(c \stackrel{*}{-} a) \cup (c \stackrel{*}{-} b) \leqslant c \stackrel{*}{-} (a \cap b)$;

(3) $a \stackrel{*}{-} b \leqslant -b \stackrel{*}{-} -a$;

(4′) $(a \stackrel{*}{-} b) \stackrel{*}{-} c = (a \stackrel{*}{-} c) \stackrel{*}{-} b$,

(4″) $a \stackrel{*}{-} (b \stackrel{*}{-} c) \geqslant (a \stackrel{*}{-} b) \cup (a \stackrel{*}{-} -c)$;

(5) $a \rightarrow b = \Omega$ *if and only if* $a \stackrel{*}{-} b = \varnothing$;

(6′) *If* $a \leqslant b$, *then the equality* $b \stackrel{*}{-} a = \varnothing$ *holds if and only if* $-a = -b$, (6″) $a \overset{*}{-} b = b \overset{*}{-} a = \varnothing$ *if and only if* $-a = -b$;

(7) *If* $a \leqslant b$ *and* $b \overset{*}{-} a = \varnothing$, *then* $c \overset{*}{-} a = c \overset{*}{-} b$ *for any* $c \geqslant b$;

(8) $-(a \cap b) \overset{*}{-} (-a \cup -b) = \varnothing$;

(9) *Let* $b \leqslant a$ *and* $a \overset{*}{-} b = c$; *if* $a \overset{*}{-} c = d$, *then* $b \leqslant d$ *and* $d \overset{*}{-} b = \varnothing$;

(10) $(a \overset{*}{-} (a \cap b)) \overset{*}{-} (a \overset{*}{-} b) = \varnothing$;

(11) $-((a \to b) \cup (a \overset{*}{-} b)) = \varnothing$.

(1′) and (1″) follow immediately from the definition of $\overset{*}{-}$ and the fact that every revatively pseudo-complemented lattice is distributive.

(2′) follows from $(c \cap -a) \cap -b = c \cap -(a \cup b)$, and (2″) from $(c \cap -a) \cup \cup (c \cap -b) = c \cap (-a \cup -b) \leqslant c \cap -(a \cap b)$.

Since $a \leqslant --a$, we have $-b \overset{*}{-} --a = -b \cap --a \geqslant -b \cap a = a \overset{*}{-} b$, and (3) is proved.

(4′) follows from (2′), and (4″) is evident.

It is known that the equality $a \to b = \Omega$ holds if and only if $a \leqslant b$: but, it is easy to see that the last relation is equivalent to $a \overset{*}{-} b = \varnothing$, which proves (5).

P r o o f o f (6′). Note that from $a \leqslant b$ it follows $-a \geqslant -b$. If $b \overset{*}{-} a = \varnothing$, then $-a \leqslant -b$, which, together with $-a \geqslant -b$, means that $-a = -b$. If $-a = -b$, then $b \cap -a = b \cap -b = \varnothing$, or, equivalently, $b \overset{*}{-} a = \varnothing$.

It is easy to see that (6″) is true.

(7) is the consequence of (6′).

P r o o f o f (8). Put $-(a \cap b) \overset{*}{-} (-a \cup -b) = c$, and prove that $c = \varnothing$; c satisfies the relations

(12) $$c \leqslant -(a \cap b),$$

(13) $$c \cap (-a \cup -b) = \varnothing.$$

From (12) it follows $c \cap --(a \cap b) = \varnothing$. But, from (13) it follows $c \cap -a = c \cap -b = \varnothing$, which implies $c \leqslant --a$, $c \leqslant --b$. If $c \cap a = \varnothing$ (or $c \cap b = \varnothing$), then it means that $c \leqslant -a$ ($c \leqslant -b$) which, together with $c \cap -a = \varnothing$ ($c \cap -b = \varnothing$), gives $c = \varnothing$. Suppose that $c \cap a \neq \varnothing$, $c \cap b \neq \varnothing$, and put $c_1 = c \cap a$; it must be (because of $c \cap (a \cap b) = \varnothing$) $c_1 \cap b = \varnothing$, and hence $c_1 \leqslant -b$. But, from $c_1 \leqslant c$ and $c \cap -b = \varnothing$ it follows $c_1 \cap -b = \varnothing$, which, together with $c_1 \leqslant -b$, gives $c_1 = \varnothing$, contrary to our assumption. The proof is completed.

P r o o f o f (9). We have $d = a \cap -c = a \cap -(a \cap -b) \geqslant a \cap (-a \cup b) = b$. Let us put $d \overset{*}{-} b = e$; then $e \leqslant d$ and $e \cap b = \varnothing$, which, respectively, means that $e \leqslant a$, $e \leqslant -c$ and $e \leqslant -b$. From the first and third of these relations it follows that $e \leqslant a \cap -b = c$, which, together with $e \leqslant -c$, means that $e = \varnothing$, as we wanted to prove.

P r o o f o f (10). We have

$$(a\overset{*}{-}(a\cap b))\overset{*}{-}(a\overset{*}{-}b)=(a\overset{*}{-}(a\overset{*}{-}b))\overset{*}{-}(a\cap b)\leqslant -(a\cap b)\overset{*}{-}-(a\overset{*}{-}(a\overset{*}{-}b))=$$
$$=-(a\cap b)\overset{*}{-}-(a\cap -(a\overset{*}{-}b))=-(a\cap b)\overset{*}{-}(-a\cup --(a\cap -b))=$$
$$=-(a\cap b)\overset{*}{-}(-a\cup -(-a\cup --b))=-(a\cap b)\overset{*}{-}(-a\cup(--a\cap -b))=$$
$$=-(a\cap b)\overset{*}{-}(-a\cup -(-a\cup b))=-(a\cap b)\overset{*}{-}-(a\cap(-a\cup b))=\varnothing,$$

from which it follows that (10) holds.

P r o o f o f (11). It is known that in pseudo-Boolean algebra $a\to b$ can be defined by $a\to b=-a\cup b$. Let us put $-((a\to b)\cup(a\overset{*}{-}b))=c$ and prove that $c=\varnothing$. By a simple transformation c can be written in the form $c=(--a\cap -b)\cap -(a\cap -b)$, which is equivalent to $c\leqslant --a\cap -b$ and $c\leqslant -(a\cap -b)$. From the last two relations it follows $c\leqslant --a$, $c\leqslant -b$, $c\cap(a\cap -b)=\varnothing$. Hence $c\cap a=\varnothing$, i.e., $c\leqslant -a$, which, together with $c\leqslant --a$, gives $c\leqslant -a\cap --a=\varnothing$. Hence $c=\varnothing$.

L e m m a 2. *The following statements are valid:*

(14) *If* $a\leqslant b\leqslant c$ *and* $b\overset{*}{-}a=c\overset{*}{-}b=\varnothing$, *then* $c\overset{*}{-}a=\varnothing$;

(15) *Let* $a\leqslant b$ *and* $b\overset{*}{-}a=\varnothing$; *if* c *is such that* $a\leqslant c\leqslant b$, *then* $c\overset{*}{-}a=b\overset{*}{-}c=\varnothing$.

P r o o f o f (14). Let $c\overset{*}{-}a=d$; then $d\leqslant c$ and $d\cap a=\varnothing$. Put $d_1=b\cap d$; since $d_1\leqslant b\overset{*}{-}a$, it must be $d_1=\varnothing$. Hence $b\cap d=\varnothing$, i.e., $d\leqslant c\overset{*}{-}b$, which implies $d=\varnothing$, as we wanted to prove.

P r o o f o f (15). It is clear that $c\overset{*}{-}a\leqslant b\overset{*}{-}a$ and $b\overset{*}{-}c\leqslant b\overset{*}{-}a$. From the hypothesis $c\overset{*}{-}a\geqslant\varnothing$ or $b\overset{*}{-}c\geqslant\varnothing$ it follows $b\overset{*}{-}a\geqslant\varnothing$, which contradicts our assumption.

Let a and b be arbitrary elements from A. We shall say that a and b are equivalent, and we shall write $a\sim b$, if $a\overset{*}{-}b=b\overset{*}{-}a=\varnothing$.

L e m m a 3. *The relation* $\sim$ *is an equivalence relation.*

P r o o f. We must prove that the relation $\sim$ is reflexive, symmetric and transitive. It is evident that $\sim$ has the first two properties. Let us show that, if $a\sim b$ and $b\sim c$, then $a\sim c$. Since $a\sim c$ is equivalent to $a\overset{*}{-}c=c\overset{*}{-}a=\varnothing$, we shall prove only the first of these equalities, because the proof of the second is analogous. Put $a\overset{*}{-}c=d$. If $d\cap b\geqslant\varnothing$, $d\cap b\neq\varnothing$, then, by reason of $d\cap b\leqslant b\overset{*}{-}c$, it follows $b\overset{*}{-}c\geqslant\varnothing$, $b\overset{*}{-}c\neq b$, which contradicts the assumption $b\sim c$. Hence $d\cap b=\varnothing$, i.e., $d\leqslant -b$. Since $d\leqslant a$, it follows that $d\leqslant a\overset{*}{-}b$, which is possible only in the case $d=\varnothing$. The proof is completed.

L e m m a 4. *Suppose that a, b, c, d are from A and that $a\sim b$, $c\sim d$. Then*

(16) $$-a\sim -b;$$

(17) $$a\cup c\sim b\cup d;$$

(18) $$a\cap c\sim b\cap d;$$

(19) $$a\rightarrow c\sim b\rightarrow d;$$

(20) $$a\overset{*}{-}c\sim b\overset{*}{-}d;$$

P r o o f o f (16). The condition $a\sim b$ is equivalent to $a\overset{*}{-}b=\varnothing$ and $b\overset{*}{-}a=\varnothing$, from which it follows $a\leqslant --b$, $-a\leqslant -b$. Hence $-a\cap --b=\varnothing$, or, equivalently, $-a\overset{*}{-}-b=\varnothing$. Similarly we can show that $-b\overset{*}{-}-a=\varnothing$, which means that $-a\sim -b$.

P r o o f o f (17). We have

$$(a\cup c)\overset{*}{-}(b\cup d)=(a\cup c)\cap -(b\cup d)=(a\cup c)\cap(-b\cap -d)=$$
$$=((a\cup c)\cap -b)\cap -d=(c\cap -b)\cap -d=(c\cap -d)\cap -b=\varnothing .$$

By similar way it can be shown that $(b\cup d)\overset{*}{-}(a\cup c)=\varnothing$.

P r o o f o f (18). We have, by reason of (6′),

(21) $$(a\cap c)\overset{*}{-}(b\cap d)=(a\cap c)\cap ---(b\cap d)=(a\cap c)\overset{*}{-}-(-b\cup -d).$$

But, from (16) it follows that $-a\sim -b$ and $-c\sim -d$, which, because of (17), means that $-a\cup -c\sim -b\cup -d$. From (21) and the last relation, for (6″), we obtain

(22) $$(a\cap c)\overset{*}{-}(b\cap d)=(a\cap c)\overset{*}{-}-(-a\cup -c)=$$
$$=(a\cap c)\overset{*}{-}(--a\cap --c)\leqslant(a\cap c)\overset{*}{-}(a\cap c)=\varnothing .$$

By similar way it can be shown that $(b\cap d)\overset{*}{-}(a\cap c)=\varnothing$.

(19) and (20) are evident consequences of (16), (17) and (18).

Denote by E the set of all equivalence-classes of A: $E=A/\sim$. The equivalence-class which is generated by a, $a\in A$, we shall denote by $|a|$. If $|a|$ and $|b|$ are arbitrary elements from E, then we shall say that $|a|$ is not greater than $|b|$, and we shall write $|a|\leqslant|b|$ or $|b|\geqslant|a|$, if $a\leqslant b$ for any $a\in|a|$ and any $b\in|b|$. The operations $\cup$, $\cap$, $\rightarrow$, $-$, $\overset{*}{-}$ in E we shall define obviously: $|a|\cup|b|=|a\cup b|$, $|a|\cap|b|=|a\cap b|$, $|a|\rightarrow|b|=|a\rightarrow b|$, $-|a|=|-a|$, $|a|\overset{*}{-}|b|=|a\overset{*}{-}b|$. From Lemma 4 it follows that all these operations are well defined.

It is easy to see that $\varepsilon=(E, \cup, \cap, -)$ is the Boolean algebra. Indeed, for any $|a|\in E$ we have $--|a|=|--a|$, but from $--a\overset{*}{-}a=\varnothing$ and $a\overset{*}{-}--a=\varnothing$, it follows $|--a|=|a|$, i.e., $--|a|=|a|$ for any $|a|$ from E.

Let us define on A the function p with the following properties:

(I) $p(a) \geqslant 0$ for each $a \in A$;

(II) $p(\Omega) = 1$;

(III) $p(a_1 \cup a_2 \cup \cdots \cup a_n) = \sum_{i=1}^{n} p(a_i)$ for each positive integer n, and any $a_1, \ldots, a_n \in A$ such that $a_i \cap a_j = \varnothing$, $i \neq j$;

(IV) if $a \leqslant b$, then $p(a) \leqslant p(b)$.

L e m m a 5. *The function p has the following properties:*

(23) $p(\varnothing) = 0$;

(24) *if* $a \leqslant b$, *then* $p(b \overset{*}{-} a) \leqslant p(b) - p(a)$;

(25) $p(a \cup b) \geqslant p(a \overset{*}{-} b) + p(b \overset{*}{-} a) + p(a \cap b)$.

From $\varnothing \cup \Omega = \Omega$ and $\varnothing \cap \Omega = \varnothing$, by reason of (II) and (III), it follows $p(\varnothing) + 1 = 1$, which gives (23).

From $(b \overset{*}{-} a) \cap a = \varnothing$ and $a \cup (b \overset{*}{-} a) \leqslant b$, by reason of (III) and (IV), it follows (24).

P r o o f o f (25). From $a \overset{*}{-} b \leqslant a$ and $b \overset{*}{-} a \leqslant b$ it follows

(26) $$(a \overset{*}{-} b) \cup (b \overset{*}{-} a) \cup (a \cap b) \leqslant a \cup b.$$

From the definition of the operation $\overset{*}{-}$ it follows $(a \overset{*}{-} b) \cap b = \varnothing$, $(b \overset{*}{-} a) \cap a = \varnothing$, and hence $(a \overset{*}{-} b) \cap (a \cap b) = \varnothing$, $(b \overset{*}{-} a) \cap (a \cap b) = \varnothing$. Also, we have $(a \overset{*}{-} b) \cap (b \overset{*}{-} a) = (a \cap -b) \cap (b \cap -a) = (a \cap b) \cap (-b \cap -a) = (a \cap b) \cap -(a \cup b) = \varnothing$, which is, together with the previous two equalities, enough that from (26), by reason of (III) and (IV), follows (25).

If $a \leqslant b$ and $b \overset{*}{-} a = \varnothing$, then the natural question is: Is there a reason for the inequality $p(a) < p(b)$? The following lemmas are in connection with this question.

L e m m a 6. *If for arbitrary two elements a,* $b \in A$, *such that* $a \leqslant b$ *and* $b \overset{*}{-} a = \varnothing$, *the equality* $p(a) = p(b)$ *is satisfied, then for any a,* $b \in A$, *such that* $a \overset{*}{-} b = b \overset{*}{-} a = \varnothing$, *the equality* $p(a) = p(b)$ *is satisfied.*

P r o o f. From (10) and the assumption of Lemma we obtain $a \overset{*}{-} (a \cap b) = b \overset{*}{-} (a \cap b) = \varnothing$, which, by reason of $a \cap b \leqslant a$, $a \cap b \leqslant b$, implies $p(a) = p(a \cap b) = p(b)$, as we wanted to prove.

L e m m a 7. *The following statements are equivalent:*

(27) *For any a,* $b \in A$, *if* $a \leqslant b$, $b \overset{*}{-} a \varnothing$, *then* $p(a) = p(b)$;

(28) *The equality* $p(a) + p(-a) = 1$ *holds for any* $a \in A$;

(29) *If a,* $b \in A$, $a \leqslant b$, *then* $p(b \overset{*}{-} a) = p(b) - p(a)$.

P r o o f. Suppose that (27) holds, and prove (29). We have $a \cup (b \overset{*}{-} a) \leqslant b$ and, by definition, $b \overset{*}{-} (a \cup (b \overset{*}{-} a)) = \varnothing$, which, by reason of (27) and (III), means that $p(b) = p(a) + p(b \overset{*}{-} a)$.

If (29) holds and $a \leqslant b$, $b \overset{*}{-} a = \varnothing$, then we have $p(b \overset{*}{-} a) = 0 = p(b) - p(a)$, which means that (27) holds.

Suppose that (29) holds and put $b = \Omega$; then we obtain $p(-a) = 1 - p(a)$, which means that (28) holds.

If (28) holds and $a \leqslant b$, $b \overset{*}{-} a = \varnothing$, then, by reason of (6′), $-a = -b$, which means that $p(a) = p(b)$.

Thus, we proved that the statements (27), (28) and (29) are equivalent.

C o r o l l a r y. (IV) *is the consequence of* (27).

If (27) holds and if a, b are arbitrary elements from A such that $a \leqslant b$, then from (27) it follows that $p(b) = p(b \overset{*}{-} a) + p(a) \geqslant p(a)$, which proves (IV).

It is easy to see that, if some of the statements from the previous Lemma holds, then the equality

$$p(a \cup b) = p(a) + p(b) - p(a \cap b) \tag{30}$$

holds for any a, $b \in A$. Namely, we have $a \cup ((a \cup b) \overset{*}{-} a) \leqslant a \cup b$ and $(a \cup b) \overset{*}{-} (a \cup ((a \cup b) \overset{*}{-} a)) = \varnothing$, which, together with (27), (III) and (1′), means that $p(a \cup b) = p(a) + p(b \overset{*}{-} a)$. Hence, by reason of (10) and (29), $p(b \overset{*}{-} a) = p(b \overset{*}{-} (a \cap b)) = p(b) - p(a \cap b)$, which, together with the previous equality, gives (30).

Suppose that the function p has the following property, which represents a generalization of the property (III):

(III′) If $a_1, a_2, \ldots \in A$ are such that $a_i \cap a_j = \varnothing$ for $i \neq j$, and $\bigcup_{i=1}^{\infty} a_i = a_1 \cup a_2 \cup \cdots$ exists, then $p\left(\bigcup_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} p(a_i)$.

L e m m a 8. *Suppose that some of the statements from Lemma 7 holds. If $a_1, a_2, \ldots$ are elements from A, such that $a_1 \leqslant a_2 \leqslant \ldots$, $\bigcup_{i=1}^{\infty} a_i$ exists, then* $p\left(\bigcup_{i=1}^{\infty} a_i\right) = \lim_{n \to \infty} p(a_n)$.

P r o o f. We have

$$a_1 \cup (a_2 \overset{*}{-} a_1) \cup (a_3 \overset{*}{-} a_2) \cup \cdots \leqslant \bigcup_{i=1}^{\infty} a_i,$$

but, it is easy to see that ($a_0 = \varnothing$)

$$\bigcup_{i=1}^{\infty} a_i \overset{*}{-} (a_1 \cup (a_2 \overset{*}{-} a_1) \cup (a_3 \overset{*}{-} a_2) \cup \cdots) =$$

$$= \bigcup_{i=1}^{\infty} (a_i \overset{*}{-} (a_1 \cup (a_2 \overset{*}{-} a_1) \cup (a_3 \overset{*}{-} a_2) \cup \cdots)) =$$

$$= \bigcup_{i=1}^{\infty} \left(a_i \stackrel{*}{-} \bigcup_{j=1}^{\infty} (a_j \stackrel{*}{-} a_{j-1})\right) \leqslant \bigcup_{i=1}^{\infty} \left(a_i \stackrel{*}{-} \bigcup_{j=1}^{i} (a_j \stackrel{*}{-} a_{j-1})\right) =$$

$$= \bigcup_{i=1}^{\infty} \left(a_i - \bigcup_{j=1}^{i} (a_j \stackrel{*}{-} a_{j-1})\right) = \bigcup_{i=1}^{\infty} \left(a_i \cap \bigcap_{j=1}^{i} -(a_j \cap -a_{j-1})\right) =$$

$$= \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{i} (a_i \cap -(a_j \cap -a_{j-1})) = \bigcup_{i=1}^{\infty} \varnothing = \varnothing,$$

which, by reason of (27), means that

$$(31) \qquad p\left(\bigcup_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} p(a_i \stackrel{*}{-} a_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} p(a_i \stackrel{*}{-} a_{i-1}) =$$

$$= \lim_{n\to\infty} p\left(\bigcup_{i=1}^{n} (a_i \stackrel{*}{-} a_{i-1})\right).$$

By reason of

$$a_n \stackrel{*}{-} \bigcup_{i=1}^{n} (a_i \stackrel{*}{-} a_{i-1}) = \varnothing,$$

from (31) it follows

$$p\left(\bigcup_{i=1}^{\infty} a_i\right) = \lim_{n\to\infty} p(a_n),$$

as we wanted to prove.

This Lemma shows that the function p, which has the additional properties (III′) and (27), is continuous with respect to the operation $\cup$. In the following example we shall see that it is not the case for the operation $\cap$.

Example. Suppose that A contains only the sets of the form $a_s = (0;\, s)$, where s is an arbitrary real number from the interval $(0;\, 1)$; put $\Omega = (0;\, 1)$ and $-a_s = \varnothing$ for each s. If we suppose that all the other operations are defined in the usual way, then A becomes the pseudo-Boolean algebra with respect to these operations. Suppose that the function p, which is defined on A, has the properties (I), (II), (III′) and (27). It is easy to see that it must be $p(a_s) = 1$ for each s. Consider the sequence $a_1, a_2, \ldots$, where $a_i = (0;\, 1/i)$, $i = 1, 2, \ldots$; we have $a_1 \geqslant a_2 \geqslant \cdots$ and $\bigcap_{i=1}^{\infty} a_i = \varnothing$, which means that $p\left(\bigcap_{i=1}^{\infty} a_i\right) = 0$, but, by reason of $p(a_i) = 1$, $i = 1, 2, \ldots$, it is also $\lim_{n\to\infty} p(a_n) = 1$. Thus $p\left(\bigcap_{i=1}^{\infty} a_i\right) < \lim_{n\to\infty} p(a_n)$.

It can be shown that from the continuity of p with respect to $\cup$ it follows the property (III′) of p. More precisely, the following statament holds.

Lemma 9. *If the statament from Lemma* 8 *is valid, then the statament* (III′) *is also valid.*

Proof. If $a_1, a_2, \ldots \in A$ are such that $a_i \cap a_j = \varnothing$ for $i \neq j$, and $\bigcup_{i=1}^{\infty} a_i$ exists, then we define the new sequence $b_1, b_2, \ldots$ by

$$b_i = \bigcup_{j=1}^{i} a_i, \qquad i = 1, 2, \ldots ;$$

this sequence satisfies all assumptions of Lemma 8, from which it follows that $p\left(\bigcup_{i=1}^{\infty} b_i\right) = \lim_{n\to\infty} p(b_n)$. But, it is clear that $\bigcup_{i=1}^{\infty} b_i = \bigcup_{i=1}^{\infty} a_i$, which, together with the previous equality, gives

$$p\left(\bigcup_{i=1}^{\infty} a_i\right) = p\left(\bigcup_{i=1}^{\infty} b_i\right) = \lim_{n\to\infty} p(b_n) = \lim_{n\to\infty} p\left(\bigcup_{i=1}^{n} a_i\right) =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} p(a_i) = \sum_{i=1}^{\infty} p(a_i),$$

as we wanted to prove.

Suppose that the function p satisfies (I), (II), (III) and (27). Then this function has the same value on all elements of A belonging to the same equivalence-class from E. Moreover, if we define the new function $\tilde{p}$ on E by

$$\tilde{p}(|a|) = p(a) \text{ for any } a \in |a|, \ |a| \in E,$$

then it is easy to see that $\tilde{p}$ is the probability function on E.

It is known that for every pseudo-Boolean algebra $\mathcal{A}$ there exists a topological Boolean algebra $\mathcal{B} = (B, \cup, \cap, ^-)$, such that $A = \sigma(B)$, where $\sigma(B)$ denotes the set of all open elements in B, [1], (the sign "$-$" denotes the complement in B). The interesting question is: if the function p is defined on A, is there any probability function P on B, such that $P(a) = p(a)$ for each $a \in A$? We shall give the answer to this question for the case when the topological Boolean algebra is formed in one special way. Namely, let $\mathcal{B} = (B, \cup, \cap, ^-)$ be the topological Boolean algebra satisfying the following conditions (see [1], p. 128):

(a) every element $b \in B$ is of the form

$$b = (\overline{a_1} \cup a_1') \cap \cdots \cap (\overline{a_n} \cup a_n'), \tag{32}$$

where $a_1, a_1', \ldots, a_n, a_n'$ are elements from A, n is arbitrary positive integer and $\overline{a_i}$ is the complement of a_i in B. (Since every distributive lattice is isomorphic to a set lattice, we can suppose that A is a lattice of subset of a set Ω, which implies that $\overline{a}$ is the ordinary set-theoretical complement of a with respect to Ω.)

(b) the interior operation in B is defined as follows: if $b \in B$ is of the form (32), then

$$Ib = (-a_1 \cup a_1') \cap \cdots \cap (-a_n \cup a_n'), \tag{33}$$

where $-a_i$ is, as above, the pseudo-complement of a_i in A. It can be shown [1] that the equality $A=\sigma(B)$ is satisfied.

Theorem. Let $\mathcal{A}$ be the pseudo-Boolean algebra and let on A the function p, with the properties (I), (II), (III) and (27) is defined. Also, let $\mathcal{B}$ be the topological Boolean algebra satisfying (a) and (b). The function P, defined by

$$P(b)=p(Ib), \qquad b\in B, \tag{34}$$

is the probability function on B, i.e., the non-negative, normed and additive function on B.

Proof. From (I) and (II) it follows that the function P is non-negative and normed. Let us prove that it is additive. Suppose that b' and b'' are arbitrary elements from B, such that $b'\cap b''=\varnothing$; these elements have forms

$$b'=(\overline{a}_1\cup a_1')\cap\cdots\cap(\overline{a}_n\cup a_n'),$$

$$b''=(\overline{b}_1\cup b_1')\cap\cdots\cap(\overline{b}_m\cup b_m'),$$

where $a_1, a_1', \ldots, a_n, a_n', b_1, b_1', \ldots, b_m, b_m'$ are from A. We must prove that

$$P(b'\cup b'')=P(b')+P(b''). \tag{35}$$

It is easy to see that $b'\cup b''$ can be represented in the form

$$b'\cup b''=(\overline{(a_1\cap b_1)}\cup(a_1'\cup b_1'))\cap\cdots\cap(\overline{(a_1\cap b_m)}\cup(a_1'\cup b_m'))\cap\cdots\cap$$

$$\cap(\overline{(a_n\cap b_1)}\cup(a_n'\cup b_1'))\cap\cdots\cap(\overline{(a_n\cap b_m)}\cup(a_n'\cup b_m')).$$

Hence, by reason of (34) and (33)

$$P(b'\cup b'')=p[(-(a_1\cap b_1)\cup(a_1'\cup b_1'))\cap\cdots\cap(-(a_n\cap b_m)\cup(a_n'\cup b_m'))].$$

It is easy to show that

$$(-(a_1\cap b_1)\cup(a_1'\cup b_1'))\cap\cdots\cap(-(a_n\cap b_m)\cup(a_n'\cup b_m'))\sim \tag{36}$$

$$\sim((-a_1\cup -b_1)\cup(a_1'\cup b_1'))\cap\cdots\cap((-a_n\cup -b_m)\cup(a_n'\cup b_m'));$$

namely, we have, for (8),

$$-(a_i\cap b_j)\sim -a_i\cup -b_j, \qquad i=1,\ldots,n, \qquad j=1,\ldots,m,$$

which, together with (17), means that

$$-(a_i\cap b_j)\cup(a_i'\cup b_j')\sim(-a_i\cup -b_j)\cup(a_i'\cup b_j'),$$

$$i=1,\ldots,n, \qquad j=1,\ldots,m,$$

from which, by reason of (18), it follows (36). From the assumption (27) and from (36) it follows

$$P(b' \cup b'') = p\,[((-a_1 \cup -b_1) \cup (a_1' \cup b_1') \cap \cdots \cap ((-a_n \cup -b_m) \cup (a_n' \cup b_m'))].$$

But, the expression in the parentheses on the right side in the last equality can be written in the following way:

$$((-a_1 \cup -b_1) \cup (a_1' \cup b_1')) \cap \cdots \cap ((-a_n \cup -b_m) \cup (a_n' \cup b_m')) =$$

$$= [(-a_1 \cup a_1') \cap \cdots \cap (-a_n \cup a_n')] \cup [(-b_1 \cup b_1') \cap \cdots \cap (-b_m \cup b_m')],$$

which, together with (III) and the assumption $b' \cap b'' = \varnothing$, means that the equality (35) is satisfied.

REFERENCE

[1] Rasiowa, H. and Sikorski, S., The Mathematics of Metamathematics, Polska Akademia Nauk, Warszawa, 1963.