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ANTI — INVERSE SEMIGROUPS

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1. In [1] two theorems about anti-regular semigroups were stated. Here we prove that every anti-regular semigroup is regular (Corollary 2.1. (i)) and in view of that fact suchlike semigroups we call anti-inverse. Theorems from [1] we get here as corollaries of our theorems 2.1 and 2.2.

2. Two elements a and b from semigroup S are mutual anti-inverses if

$$aba=b \quad \text{and} \quad bab=a.$$

Definition 2.1. A semigroup S will be called anti-inverse if every element from S has its anti-inverse element in S.

Example 2.1. By following tables

1)

	a	b
a	a	a
b	b	b

2)

	a	b
a	a	b
b	b	b

3)

	a	b
a	a	b
b	b	a

some anti-inverse semigroups are given. Thus in table 3) anti-inverse elements for a are a and b, for b are a and b.

Let $\mathcal{A}$ denote the class of anti-inverse semigroups.

Theorem 2.1. *Let S be a semigroup. Then*

$$S\in\mathcal{A} \Leftrightarrow (\forall\, x\in S)\,(\exists\, y\in S)\,(x^2=y^2,\; yx=x^3y,\; x^5=x).$$

P r o o f. Let $S \in \mathcal{A}$. Then for each $x \in S$ there is its anti-inverse element $y \in S$, so

$$x^2 = xx = (yxy)x = y(xyx) = yy = y^2 \tag{2.1}$$

$$yx = (xyx)(yxy) = x(yxy)xy = xxxy = x^3 y. \tag{2.2}$$

From (2.1) and (2.2) we have

$$x = yxy = x^3 yy = x^3 y^2 = x^5 \tag{2.3}$$

for every $x \in S$.

Conversely, let for each $x \in S$, there is $y \in S$ which satisfies the conditions of the theorem. Then we have

$$xyx = xx^3 y = x^4 y = y^5 = y \tag{2.4}$$

$$yxy = x^3 yy = x^3 y^2 = x^5 = x. \tag{2.5}$$

It follows from (2.4) and (2.5) that x and y are anti-inverse elements. This completes the proof of the theorem.

C o r o l l a r y 2.1. (i) *Every anti-inverse semigroup S is a regular semigroup.*

(ii) *Each element in S has its own unity.*

(iii) *Anti-inverse elements from S have the same unity.*

(iv) *If $x^2 = e_x$ (e_x is x own unity), then the element x is permutable with each of his anti-inverse elements.*

(v) *If x and y are mutual anti-inverse elements then $x^2 y = yx^2$ and $xy^2 = y^2 x$.*

(vi) *If for $x \in S$ an anti-inverse element is $y \in S$ then so are also: xy, x^2y, x^3y.*

(vii) *Every anti-inverse semigroup S is an intra-regular semigroup.*

P r o o f. (i) In an anti-inverse semigroup S is $x^5 = x$, for each x in S, that is $xx^3 x = x$, so we have that for each $x \in S$ there is $a = x^3$ such $xax = x$. Thus, S is a regular semigroup.

(ii) From $x^5 = x$ for each x in S we have $(\forall x \in S)\ (x^4 = e_x)$

(iii) Let x and y be mutual anti-inverse elements, then $x^2 = y^2$ which yields $x^4 = y^4$, i.e. $e_x = e_y$.

(iv) If y is anti-inverse element for x then $yx = x^3 y = xy$.

(v) If x and y are mutual anti-inverse elements then we have $x^2 y = y^3 = yy^2 = yx^2$. Similarly, we get the other relation.

(vi) Let x and y be anti-inverse elements. Then, for $s = 1, 2, 3$ we have

$$(x^s y)^2 = x^s yx^s y = y^2 = x^2$$

$$(x^s y)x = x^s(yx) = x^s x^3 y = x^3(x^s y)$$

which means that x and $x^s y$ ($s = 1, 2, 3$) are mutual anti-inverse elements.

(vii) Since $x^2 x^2 x = x$ for every $x \in S$, S is an intra-regular semigroup (for the definition intra-regular semigroup see [2]).

Theorem 2.2. *Let S be a semigroup. Then*

$$S \in \mathcal{A} \Leftrightarrow (\forall x \in S)(\exists y \in S)(x^2 = y^2,\ x^2 = (xy)^2,\ x^5 = x)$$

Proof. Let $S \in \mathcal{A}$. Then, by Theorem 2.1 for every $x \in S$ we have $x^5 = x$ and for each y which is anti-inverse for x, we have $x^2 = y^2$. Now let us prove that $(xy)^2 = x^2$. For anti-inverse elements x and y we have $xyx = y$, hence by multiplying this equality by y from the right, we get $xyxy = y^2 = x^2$.

Conversely, from $(xy)^2 = y^2$ by multiplying this equality by y^3 from the right, we have $xyxy^4 = y^5$, whence, by using $x^2 = y^2$ we get $xyx^5 = y^5$, and, by using $(\forall x \in S)(x^5 = x)$ we have $xyx = y$. Similarly, from $xyxy = x^2$ we have $yxy = x$, so x and y are mutual anti-inverse elements.

This completes the proof of the theorem.

By Table 1) in the example 2.1. anti-inverse semigroup which is not an inverse semigroup is given. By Table 2) anti-inverse semigroup which is an inverse semigroup but not a group is given. By Table 3) an example of a group which is an anti-inverse semigroup is given. Consequently the class of anti-inverse semigroups $\mathcal{A}$ does not include the class of inverse semigroups, nor the class of groups, but the intersections of those classes are not empty.

Let A_a denote the set of all anti-inverse elements of the element a of the anti-inverse semigroup S.

Theorem 2.3.

$$a \notin A_a \Rightarrow (\forall x, y)(x, y \in A_a \Rightarrow xy \notin A_a).$$

Proof. Suppose that this is not true, i.e. $a \notin A_a$, $x, y \in A_a$ and $xy \in A_a$. Then

$$a^3 = aaa = (xax)(yay)a = x(axya)ya = xxyya = a^5 = a.$$

This means that the element a is its own anti-inverse i.e. $a \in A_a$ contradiction.

Theorem 2.4. *Let $x, y \in A_a$. Then*

$$a \in A_a \text{ and } xy = yx \Leftrightarrow xy \in A_a.$$

Proof. Let $a \in A_a$ and $xy = yx$, then by Corollary 2.1. (iv) we have

$$a(xy)a = axay = xy \tag{2.6}$$

$$(xy)a(xy) = xyayx = a. \tag{2.7}$$

From (2.6) and (2.7) we have $xy \in A_a$.

Conversely, let $xy \in A_a$, i.e. $a(xy)a = xy$ and $(xy)a(xy) = a$. Multiplying the first of these relations by y from the right we get

$$axyay = xy^2 \tag{2.8}$$

Since $a^2 = x^2 = y^2$, from (2.8) we have

$$axa = x^3. \tag{2.9}$$

From (2.9) we have $x^3 = x$, as a and x are mutually anti-inverse, whence, multiplying that relation by x

$$x^4 = x^2 = e_x. \tag{2.10}$$

From (2.10) and using that the squares of the anti-inverse elements are equal, we have

$$e_x = x^2 = a^2. \tag{2.11}$$

As we have $e_x = e_a$, from (2.11) we get $a^2 = e_a$, so $a^3 = a$, i.e. $a \in A_a$.

As we have

$$(xy)^2 = a^2 = e_a \tag{2.12}$$

and elements a, x, y, xy have the same unity, then multiplying (2.12) by yx we get $xy = yx$.

This proves the theorem.

Theorem 2.5. *Let S be anti-inverse semigroup. Then*

$$(\forall x \in S)\,(A_x = A_{x^3})$$

Proof. Let $a \in A_x$, then

$$ax^3a = axxxa = ax(axa)xa = (axa)x(axa) = xxx = x^3. \tag{2.13}$$

$$x^3ax^3 = a. \tag{2.14}$$

From (2.13) and (2.14) we have that a and x^3 are mutuall anti-inverse elements i.e. $a \in A_{x^3}$, So

$$A_x \subset A_{x^3}. \tag{2.15}$$

Conversely, let $a \in A_{x^3}$, then $a^2 = (x^3)^2 = x^2$, and we have

$$axa = (x^3ax^3)xa = x^3ax^4a = x^3aa^4a = x^3a^2 = x^5 = x \tag{2.16}$$

$$xax = x(x^3ax^3) = x^4ax^4 = a^4aa^4 = a \tag{2.17}$$

So

$$A_{x^3} \subset A_x. \tag{2.18}$$

From (2.15) and (2.18) the theorem follows.

The next two theorems are cited in [1]. Here are their statements.

Theorem *Let S be a semigroup. Each element of S has a unique anti--inverse element in S if and only if S is an idempotent semigroup (band).*

Theorem *Let S be a semigroup. Any two elements of S are anti-inverses if and only if S is an abelian group in which each element is its own (group) inverse.*

Proof of the first theorem. Sufficiency. Let S be an idempotent semigroup, then each element x of S is its own anti-inverse. Let us suppose that for x anti-inverse element is some $y \neq x$. Then $x = x^2 = y^2$ (Theorem 2.1.) $= y$, thus $x = y$ contradiction.

Necessity. if y is the unique anti-inverse for x, then in virtue of the Corollary 2.1. (vi) xy is also anti-inverse of x and so $y = xy$. Since $x^2 = y^2 = (xy)^2$ (Theorem 2.2.), then multiplying $y = xy$ by y from the right we get $y^2 = xy^2$ i.e. $x^2 = x^3$ which by multiplication with x^3 yields $x^5 = x^5 x$ what in view of $x^5 = x$ (Theorem 2.2.) yields the requested idempotency $x = x^2$.

Proof of the second theorem. Let S be a semigroup in which any two elements are mutually anti-inverse. Then, by Theorem 2.1. we have

$$(\forall x)(\forall y)(x^2 = y^2,\ yx = x^3 y,\ x^5 = x)$$

whence, for $x = y$ we have $x^2 = x^4$ i.e. $x^3 = x$, so

$$(\forall x)(\forall y)(xy = yx).$$

Since $(\forall x \in S)(x^4 = e)$ (e is the unity, Corollary 2.1. (iii)) we have $x^3 = x^{-1}$ i.e. $x = x^{-1}$.

Conversely, as S is an abelian group in which $x = x^{-1}$, we have

$$(\forall x \in S)(x^2 = e)$$

where e is the unity of the group S. Then

$$(\forall x)(\forall y)(x^2 = y^2 = (xy)^2 = e)$$

and by Theorem 2.2. the proof is completed.

3. Let P be a non empty subset of a semigroup S. Let $[P]$ denote the subsemigroup of S generated by the set P.

Theorem 3.1. *Let S be an anti-inverse semigroup and $a \in S$. Then for each subset $I_a \subset A_a$, $GI_a := [a \cup I_a]$ is a subgroup of S.*

Proof. If $I_a = \varnothing$ and $a \notin A_a$, then $GI_a = [a]$ is the cyclic group of order 4.

Let $I_a \neq \varnothing$. For any element x of GI_a we have $x = a_1 a_2 \cdots a_n$, for some $a_i \in a \cup I_a$. Let us prove that $e_x = x^4 = e_a$.

First let us prove the following equality

$$(a_1 a_2 \cdots a_n)^2 = a_n a_{n-1} \cdots a_1 (a_1 a_2 \cdots a_n)^4 a_n a_{n-1} \cdots a_1 \tag{3.1}$$

where a_i from I_a $(i=1, 2, \ldots, n)$.

Starting from the right side of (3.1) we get

$$a_n a_{n-1} \cdots a_1 (a_1 a_2 \cdots a_n)^4 a_n a_{n-1} \cdots a_1 =$$

$$= a_n a_{n-1} \cdots a_1 (a_1 a_2 \cdots a_n)^2 a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n a_n a_{n-1} \cdots a_1$$

$$= a_n a_{n-1} \cdots a_1 (a_1 a_2 \cdots a_n)^2 a_1 a_2 \cdots a_n a^{2n}$$

(because $a_i a_i = a^2$ and $a^2 a_i = a_i a^2$, $i=1, 2, \ldots, n$; Corollary 2.1. (v)).

$$= a_n a_{n-1} \cdots a_1 a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n a^{2n}$$

$$= a^{2n} a^{2n} (a_1 a_2 \cdots a_n)^2$$

$$= a^{4n} (a_1 a_2 \cdots a_n)^2$$

$$= e_a a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n$$

$$= (a_1 a_2 \cdots a_n)^2 \quad (\text{as } e_a a_1 = a_1).$$

This proves the relation (3.1).

Multiplying (3.1) by $(a_1 a_2 \cdots a_n)^2$ from the right we get

$$(a_1 a_2 \cdots a_n)^4 =$$

$$= a_n a_{n-1} \cdots a_1 (a_1 a_2 \cdots a_n)^4 a_n a_{n-1} \cdots a_1 (a_1 a_2 \cdots a_n)^2$$

$$= a_n a_{n-1} \cdots a_1 (a_1 a_2 \cdots a_n)^4 a_1 a_2 \cdots a_n a^{2n}$$

$$= a_n a_{n-1} \cdots a_1 (a_1 a_2 \cdots a_n)^5 a^{2n}$$

$$= a_n a_{n-1} \cdots a_1 a_1 a_2 \cdots a_n a^{2n}$$

$$= a^{2n} a^{2n}$$

$$= a^{4n}$$

$$= e_a.$$

So all elements from GI_a have the same unity. Let us denote it by e.

For each element $x \in GI_a$ we have $xx^3 = e$, hence we get that x^3 is inverse for x i.e. $x^{-1} = x^3$.

This proves that GI_a is a group.

Corollary 3.1. (i) *If the set I_a has exactly one member and $a \notin A_a$ then GI_a is the quaternion group.*

(ii) *If* $a \in I_a$ *and* I_a *has exactly two members i.e.* $I_a = \{a, b\}$ *then* GI_a *is the Klein group or the cyclic group of order* 2.

P r o o f. (i) Let $I_a = \{b\}$, then $a^2 = b^2$, $a^4 = e$, $ba = a^3 b$.

(ii) Similarly, we have $a^2 = b^2 = (ab)^2 = e$. If $a \neq e$, then we get the Klein group, if $a = e$ we get the cyclic group of order 2.

It follows from the Theorem 3.1. that every anti-inverse semigroup S is covered by groups i.e.

$$S = \bigcup_{a \in S} GI_a.$$

For an arbitrary element $a \in S$ and an arbitrary subset $I_a \subset A_a$ the group GI_a is not anti-inverse semigroup.

For example, if $I_a = \varnothing$ and $a^2 \neq e_a$, the group GI_a is the cyclic group of order 4 and is not anti-inverse; element a has not anti-inverse of its own. For I_a in Corollary 3.1. (i) and (ii) the groups GI_a are anti-inverse. Really, for the cyclic group of order 2, $\{e, a\}$ we have $A_e = A_a = \{e, a\}$. For the Klein group $\{e, a, b, ab\}$ we have $A_e = A_a = A_b = A_{ab} = \{e, a, b, ab\}$. For the quaternion group $\{e, a, a^2, a^3, b, ab, a^2 b, a^3 b\}$ we have $A_e = A_{a^2} = \{e, a^2\}$ $A_a = A_{a^3} = \{b, ab, a^2 b, a^3 b\}$, $A_b = A_{a^2 b} = \{a, ab, a^3, a^3 b\}$ $A_{ab} = A_{a^3 b} = \{a, b, a^3, a^2 b\}$.

L e m m a 3.1. *If in a group* GI_a *for every element of the form* $b_1 b_2 \cdots b_{2n}$ *where* $b_i \neq b_{i+1}$, $b_k \in I_a (\neq \varnothing)$, $k = 1, 2, \ldots 2n$; $n = 1, 2, \ldots$ *is valid*

$$\text{(U)} \qquad (b_1 b_2 \cdots b_{2n})^2 = a^2 \wedge (\exists\, i \in \{1, \ldots, 2n-1\}) \, ((b_1 b_2 \cdots b_i)^2 = a^2 \vee (b_{i+1} \cdots b_{2n})^2 = a^2)$$

then the element $b_1 b_2 \cdots b_{2n}$ *has its anti-inverse element.*

P r o o f. Let the condition (U) be satisfied for the element $x = b_1 b_2 \cdots b_{2n}$. Then $y = b_1 b_2 \cdots b_i$ (where i is the one of the existing from the condition (U)) is anti-inverse for x.

Really, $y^2 = a^2 = x^2$,

$$\begin{aligned}
(yx)^2 &= (b_1 b_2 \cdots b_i b_1 b_2 \cdots b_{2n})^2 \\
&= ((b_1 b_2 \cdots b_i)^2 b_{i+1} \cdots b_{2n})^2 \\
&= (a^2 b_{i+1} \cdots b_{2n})^2 \\
&= a^2 b_{i+1} \cdots b_{2n} a^2 b_{i+1} \cdots b_{2n} \qquad \text{(Corollary 2.1. (v))} \\
&= (b_{i+1} \cdots b_{2n})^2 = a^2 = x^2.
\end{aligned}$$

Accordingly, y and x are anti-inverse elements (Theorem 2.2.).

The next theorem gives a sufficient condition for $GI_a (I_a \neq \varnothing)$ to be anti-inverse group.

Theorem 3.2.

$$(\forall\, x \in GI_a)\,(x^2 = e \vee (x^2 = a^2 \wedge (U))) \;\Rightarrow\; GI_a \in \mathcal{A}$$

Proof. An arbitrary element $x \in GI_a$ is of the form

$$x = a^s\, b_{i_1}^{\delta_1}\, b_{i_2}^{\delta_2} \cdots b_{i_k}^{\delta_k} \;(b_{ij} \in I_a;\; s = 1, 2, 3, 4;\; \delta_i = 0, 1)$$

because $b_r\, a = a^3\, b_r$ for every $b_r \in I_a\,(b_r \neq a)$.

Let us prove that for each such x there is its anti-inverse element.

If $x^2 = e$, then x is its own anti-inverse element, for $xxx = x^2\, x = x$.

If $x^2 = a^2$, then for $\delta_i = 0\,(i = 1, 2, \ldots k)$ the element x is of the form

$$x = a \quad \text{or} \quad x = a^3$$

so the anti-inverse for x in that case is any of the element $b_j \in I_a$, because of $A_a = A_{a^3}$ (Theorem 2.5.).

In the case when at least one $\delta_i \neq 0$ and $s = 0$, without loss of generality we can take that

$$x = b_1\, b_2 \cdots b_n.$$

We distinguish two cases.

Case 1. $n = 2m - 1$. Then the anti-inverse for x is a.

Really,

$$axa = ab_1\, b_2 \cdots b_{2m-1}\, a = aa^3\, b_1\, b_2 \cdots b_{2m-1} = x$$

$$xax = b_1\, b_2 \cdots b_{2m-1}\, ab_1\, b_2 \cdots b_{2m-1} = a^3\,(b_1\, b_2 \cdots b_{2m-1})^2 = a^3\, a^2 = a.$$

As in this case for the element a anti-inverse is x, so for the element a anti-inverse are also elements ax, $a^2 x$, $a^3 x$ (Corollary 2.1. (vi)).

Consequently, for the element of the form

$$a^s\, b_1\, b_2 \cdots b_{2m-1}\; (s = 1, 2, 3, 4)$$

the anti-inverse is a.

Case 2. $n = 2m$. Then we have

$$x^2 = (b_1\, b_2 \cdots b_{2m})^2 = a^2$$

and as the condition (U) is satisfied by Lemma 3.1. x has anti-inverse element $y = b_1\, b_2 \cdots b_i$ for some $i < 2m$.

As for y anti-inverse is x by Corollary 2.1. (vi) anti-inverse for y is $y^2 x$, too, i.e. $a^2 x$ (for $y^2 = a^2$).

Consequently, the elements of the form

$$b_1\, b_2 \cdots b_{2m} \quad \text{and} \quad a^2\, b_1\, b_2 \cdots b_{2m}$$

have the anti-inverse element $y = b_1\, b_2 \cdots b_i$.

It remains to prove that the elements of the form

$$ab_1 b_2 \cdots b_{2m} \quad \text{and} \quad a^3 b_1 b_2 \cdots b_{2m}$$

have their anti-inverse elements.

Since,

$$(ab_1 b_2 \cdots b_{2m})^2 = ab_1 b_2 \cdots b_{2m} ab_1 b_2 \cdots b_{2m} = a^2 (b_1 b_2 \cdots b_{2m})^2 = a^4 = e$$

the element $ab_1 b_2 \cdots b_{2m}$ is its own anti-inverse.

Similarly we get that $a^3 b_1 b_2 \cdots b_{2m}$ is its own anti-inverse.

Consequently, for an arbitrary $x \in GI_a$ we have anti-inverse element.

This proves the theorem.

Corollary 3.2. *If* $I_a = \{b_1, b_2\}$ *then the sufficient condition for* $GI_a \in \mathcal{A}$ is

$$(\forall x \in GI_a)\,(x^2 = e \vee x^2 = a^2).$$

Proof. An arbitrary element $x \in GI_a$ is of the form

$$x = a^s b_1^{\delta_1} b_2^{\delta_2} \ (s = 1, 2, 3, 4;\ \delta_i = 0, 1)$$

because $b_1 b_2 = b_2 b_1$ or $b_1 b_2 = a^2 b_2 b_1$ for $(b_1 b_2)^2 = e$ or $(b_1 b_2)^2 = a^2$. In the case $(b_1 b_2)^2 = a^2$ the condition (U) is always satisfied. We can take $i = 1$.

Theorem 3.3. *Let G be a group. Then*

$$G \in \mathcal{A} \Leftrightarrow (\forall x \in G)\,(\exists y \in G)\,([\{x, y\}] \in \mathcal{A}).$$

Proof. Let $G \in \mathcal{A}$. Then for an arbitrary $x \in G$, there is its anti-inverse element $y \in G$, i.e.

$$x^2 = y^2 = (xy)^2, \quad x^5 = x,$$

and for $x^2 \neq e$

(3.2) $$[\{x, y\}] = \{e, x, x^2, x^3, y, xy, x, x^2 y, x^3 y\}$$

is the quaternion group.

For $x^2 = e$ and $x \neq y$ $(x, y \notin \{e\})$

(3.3) $$[\}x, y\}] = \{e, x, y, xy\}$$

is the Klein group.

For $x^2 = e$ and $x = y$ $(x \neq e)$

(3.4) $$[\{x, y\}] = \{e, x\}$$

is the cyclic group of order 2.

If $x = e$, then e is its own anti-inverse element

(3.5) $$[\{x, y\}] = \{e\}.$$

Groups (3.2), (3.3), (3.4) and (3.5) are anti-inverse (Corollary 3.1. (i) and (ii)).

Conversely, as we have $[\{x, y\}] \in \mathcal{A}$, then for each $x \in G$ there is its anti-inverse element in $[\{x, y\}] \subset G$, and consequently in G.

This completes the proof of the theorem.

Finite groups which are anti-inverse are p-group. In fact in such groups every element is of order 1 or 2 or 4, so every such a group is of order 2^n $(n=0, 1, 2, \dots)$ [3].

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