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A FIXED POINT THEOREM IN ORBITALLY COMPLETE METRIC SPACES*

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Let T be a selfmapping of a metric space (X, d). According to Ćirić [1], we say T is orbitally continuous iff $\lim_{i\to\infty} T^{n_i}x = u \in X$ implies $Tu = \lim_{i\to\infty} TT^{n_i}x$ and X is T-orbitally complete iff every Cauchy sequence of the form $\{T^{n_i}x\}_{i=1}^{\infty}$ converges in X.

Recently, Ćirić [5] proved the following result:

Theorem 0. *Let (X, d) be a metric space and T a selfmapping of X. If X is T-orbitally complete and T is an orbitally continuous map which satisfies*

$$(*) \qquad d(Tx, Ty) < q \cdot \max\{d(x, y),\ (d(x, y))^{-1} d(x, Tx)\, d(y, Ty),$$
$$a(x, y)\, d(x, Ty)\, d(y, Tx)\}$$

for all x, y in X, $x \neq y$ and $q < 1$, where $a(x, y)$ is a nonnegative real function, then for each x in X $\lim_{n\to\infty} T^n x = u_x \in X$ and $Tu_x = u_x$. If in addition $a(x, y) \leqslant (d(x, y))^{-1}$, then T has a unique fixed point.

The purpose of this paper is to improve Ćirić's result to a more general case. For related results, we refer to Ćirić [2], [3], [4].

Let R^+ denote the set of nonnegative real numbers. Let H denote a family of mappings such that $h \in H$, $h: (R^+)^3 \to R^+$ and h is upper semicontinuous and nondecreasing in each coordinate variable. Also let $g(t) = h(t, t, t)$ where $g: R^+ \to R^+$.

The following lemma is due to Singh and Meade [6].

Lemma. *For every $t > 0$, $g(t) < t$ if and only if $\lim_{n\to\infty} g^n(t) = 0$.*

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Theorem 1. *Let T be a selfmapping of a metric space (X, d). Suppose that there is an $h \in H$ such that for all x, y in X, $x \neq y$*

$$\text{(C)} \qquad d(Tx, Ty) \leqslant h(d(x, y),\ (d(x, y))^{-1} d(x, Tx)\, d(y, Ty),\ a(x, y)\, d(x, Ty)\, d(y, Tx))$$

where $a(x, y)$ is a nonnegative real function and h satisfies $h(t, t, t) < t$ for all $t > 0$. If X is T-orbitally complete and T is an orbitally continuous mapping, then for each x in X, $\lim_{n\to\infty} T^n x = u_x \in X$ and $Tu_x = u_x$. If in addition $a(x, y) \leqslant (d(x, y))^{-1}$, then T has a unique fixed point.

Proof Let x be any point of X and assume that $Tx \neq x$. Then by (C)

$$d(Tx, T^2 x) \leqslant h(d(x, Tx),\ (d(x, Tx))^{-1} d(x, Tx)\, d(Tx, T^2 x),\ 0) = h(d(x, Tx),\ d(Tx, T^2 x),\ 0).$$

Assume $d(x, Tx) < d(Tx, T^2 x)$. Thus

$$d(Tx, T^2 x) \leqslant h(d(Tx, T^2 x),\ d(Tx, T^2 x),\ d(Tx, T^2 x)) < d(Tx\ T^2 x),$$

a contradiction. This contradiction proves that $d(Tx, T^2 x) \leqslant d(x, Tx)$. Since $d(Tx, T^2 x) = 0 \leqslant d(x, Tx)$ for the case $Tx = x$, we have

$$d(Tx, T^2 x) \leqslant d(x, Tx).$$

Similarly, $d(T^2 x, T^3 x) \leqslant g(d(Tx, T^2 x)) \leqslant g^2(d(x, Tx))$ and in general

$$d(T^n x, T^{n+1} x) \leqslant g^n(d(x, Tx)).$$

Since $\lim_{n\to\infty} g^n(t) = 0$ for $t > 0$, therefore

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0.$$

Employing the method as described in [6], we can prove that $\{T^n x\}$ is a Cauchy sequence. By the orbital completeness of X there exists some u_x in X such that

$$\lim_{n\to\infty} T^n x = u_x.$$

Since T is orbitally continuous, we have

$$Tu_x = \lim_{n\to\infty} T^{n+1} x = u_x.$$

Let $a(x, y) \leqslant (d(x, y))^{-1}$ and suppose that $u = Tu$, $v = Tv$ and $u \neq v$. Then

$$\begin{aligned} d(u, v) = d(Tu, Tv) &\leqslant h(d(u, v),\ (d(u, v))^{-1} d(u, u)\, d(v, v),\ a(u, v)\, d(u, Tv)\, d(v, Tu)) \\ &\leqslant h(d(u, v),\ 0,\ (d(u, v))^{-1} d(u, v)\, d(u, v)) \\ &< g(d(u, v)) \\ &< d(u, v), \end{aligned}$$

a contradiction. This contradiction proves our Theorem.

We state a simple example of a mapping T that satisfies (C) but not (*) for any value of $q<1$.

E x a m p l e. Let $X=[0, \infty)$ with $d(x, y)=|x-y|$. Define two mappings $T: X \rightarrow X$ and $h:(R^{+})^{3} \rightarrow R^{+}$ by

$$Tx=x(1+x)^{-1}$$

and

$$h(x, y, z)=x(1+x)^{-1}$$

for x, y, z in R^{+}. We see easily that h satisfies all the conditions of Theorem 1. Furthermore, for any x, y in X, $x \neq y$,

$$d(Tx, Ty)=\frac{|x-y|}{1+x+y+xy} \leqslant \frac{|x-y|}{1+|x-y|}=h(d(x, y),$$

$$(d(x, y))^{-1} d(x, Tx) d(y, Ty), 0),$$

where $a(x, y)=0$. Hence (C) holds. Since

$$T^{n} x=x(1+nx)^{-1} \rightarrow 0 \text { as } n \rightarrow \infty$$

implies

$$T 0=\lim_{n \rightarrow \infty} T T^{n} x,$$

T is orbitally continuous and X is T-orbitally complete. It follows Theorem 1 that T has a unique fixed point in X. In fact, $T0=0$ is the unique fixed point of T in X. However, T does not satisfy (*), for otherwise there is a $q<1$ such that for all x in X, $x \neq 0$.

$$x(1+x)^{-1}=d(T 0, Tx)<q \cdot \max \{x, 0, 0\}=qx$$

Hence $(1+x)^{-1}<q$ for any x in X, $x \neq 0$. This is clearly impossible. Thus T does not satisfy (*) for any value of $q<1$.

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REFERENCES

[1] Ć i r i ć, Lj., *On contraction type mappings*, Math. Balkanica 1 (1971), 52—57.

[2] Ć i r i ć, Lj., *Fixed and periodic points almost contractive operators*, Math. Balkanica 3 (1973), 33—44.

[3] Ć i r i ć, Lj., *Fixed points for generalized multi-valued contractions*, Mat. vesnik 9 (24), 1972, 265—272.

[4] Ć i r i ć, Lj., *On some maps with a nonunique fixed point*, Publ. Inst. Math., 17 (31), 1974, 52—58.

[5] Ć i r i ć, Lj., *A certain class of maps and fixed point theorems*, Publ. Inst. Math., 20 (34), 1976, 73—77.

[6] S i n g h, S. P. and M e a d e, B. A., *On common fixed point theorems*, Bull. Austral. Math. Soc., 16 (1977), 49—53.

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