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PROPERTIES OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $xx''-kx'^2+f(x)=0$

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Summary. A continuation of the work of L. Roth and generalizations of this work are given and applied to complete the study of a class of previously identified differential equations.

1. Introduction.

The focus of this paper is on second order differential equations wherein there is an x'^2 term. All of the equations of Sections 2, 3, 4 are autonomous and sometimes called second order differential equations of polynomial class. Much of the work of P. Painlevé concerned this class of equations and the Painlevé transcendents come from that work. In this paper we fill a lacuna of L. Roth [9] from which we are led to the solutions of specific equations for which there is only a partial treatment by prior authors.

In Section 5 we add to the literature of equations containing an x'^2 term and integrable in closed form. In that section the equations are not all of polynomial class.

In earlier papers [10, 11, 12] we have treated non-singular second order equations with quadratic damping and cited applications of these equations.

2. A problem of L. Roth. Certain differential equations of the form

(1) $$Ax''+Bx'^2\times Cx'+D=0,\quad x'=dx/dt$$

wherein A, B, C, D are polynomials in x, were studied by L. Roth [9]. The coefficients were linear except that D could be quadratic and the approach was that of P. Painlevé [7].

To treat (1) it was reduced to

(2) $$x(x''+ax'+bx+c)+l(x'-m)(x'-n)=0$$

Roth's results are valid if $lm=(l+1)n$, $an+c=0$ and so do not include the equation

(3) $$xx''-kx'^2+qx=0$$

unless $q=0$ in (3).

In this, and the following section, we consider the exceptional equation (3) and, more generally,

(4) $$xx'' - kx'^2 + f(x) = 0$$

for certain $k > 0$ and continuous functions $f(x) \not\equiv 0$ that assure some periodic solutions. Even for equation (3) all solutions are not periodic.

The special case of (3) wherein $k = 1$ may be solved for all real numbers q to reveal the behavior of all solutions in the phase plane. If in

(5) $$xx'' - x'^2 + qx = 0$$

one lets $x' = p$, $x'' = p\, dp/dx$, one secures the Bernoulli equation

$$xp \frac{dp}{dx} - p^2 + qx = 0$$

which, upon solving by elementary methods yields solutions

(6) $$x = \frac{q}{2c_1^2} [\cos h(\sqrt{2}\, c_1 (t + c_2)) - 1]$$

which are clearly not periodic.

On the other hand, we can exhibit periodic solutions of (5). Since (5) may be written as $x(x'' + q) = x'x'$ we seek an integrating function $m(t)$ such that

(7) $$x'' + q = mx'$$

(8) $$x' = mx$$

Assuming that $m(t)$ exists with a second derivative, we use (8) to eliminate x' from (7). This gives the equation

(9) $$x = -q/m'$$

We now seek a tractable equation involving $m(t)$ so as to secure solutions of (5) by putting $m(t)$ into (9). If we combine (8) with (9), we secure $m'' + mm' = 0$ from which we have

$$m(t) = -c_1 \tan \frac{c_1 (t + c_2)}{2}$$

Using (9) we secure the periodic solutions

(10) $$x = \frac{2q}{c_1^2} \cos^2 \frac{c_1 (t + c_2)}{2}$$

of (5). The curves of the (x, x')-plane corresponding to (10) are ellipses while the curves corresponding to (6) are hyperbolas. The separatrix for these two families of solutions is the parabola $x'^2 = 2qx$ in the (x, x')-plane. The ellipses are inside the parabola. Solutions at (0, 0) are, of course, not unique.

3. The Roth problem solved and generalized. We now consider equation (3) which we write as the system

$$x' = y$$

$$y' = \frac{ky^2 - qx}{x}, \quad x \neq 0$$

from which we have

$$\frac{dy}{dx} = \frac{ky^2 - gx}{yx}$$

This equation is the Bernoulli equation

$$\frac{dy}{dx} - \frac{ky}{x} = -qy^{-1}$$

If we set $z = y^2$ and then solve the resulting equation for z, we secure

(11)
$$y^2 = z = \frac{2qx}{2k-1} + cx^{2k}, \quad k \neq 1/2$$

and

(12)
$$y^2 = z = -2qx \log |x| + cx, \quad k = 1/2$$

Assuming $q \neq 0$ (the case $q = 0$ is trivial), regardless of the sign of q and c, the graph of (12) has a cycle in the (x, y)-plane and there is a corresponding periodic solution. For example, if $q > 0$, then the cycle is in the half-plane $x \geqslant 0$ between $x = 0$ and $x = \exp(c/2q)$. Solutions are not all periodic, however, as may be seen from the graph of (12) in the half-plane $x \geqslant 0$ in case $q < 0$.

To avoid restricting k to the rational numbers we seek periodic solutions of (11) for the half-plane $x \geqslant 0$. If $k > 1/2$ and $q > 0$, then for any $c > 0$ there is a cycle in $x \geqslant 0$ between $x = 0$ and

$$x = \left(\frac{2q}{c(1-2k)}\right)^{1/(2k-1)}$$

If $q > 0$, there will not be a cycle unless x^{2k} is meaningful for $x < 0$. Similarly, for $0 < k < 1/2$, if $q > 0$ one may now take any $c > 0$ to secure a cycle in the half-plane $x \geqslant 0$. If $k < 0$, there are no periodic solutions.

Thus we have proved the following theorem.

Theorem 1. *If in* (3), $k > 0$ *and* $q > 0$, *then the equation has an infinite number of periodic solutions and an infinite number of non-periodic solutions.*

The arguments given above may be used to examine the more general equation (4) for non-linear, but continuous, $f(x)$. Since it is not our aim to be definitive in this case we will only give a sufficient condition for the existence of periodic solutions.

From equation (4) we secure, as before, the system

$$x' = y, \; y' = \frac{ky^2 - f(x)}{x} \tag{13}$$

from which one secures

$$y^2 = x^{2k} \left[c - \int_0^x 2 f(u) u^{-2k-1} du \right] \tag{14}$$

corresponding to (11) and (12). Again, considering only the half-plane $x \geqslant 0$ to avoid restricting k, it is clear that there will be a periodic solution whenever c may be selected to cause

$$\varphi(x) = c - \int_0^x 2 f(u) u^{-2k-1} du \tag{15}$$

to satisfy

$$\varphi(x) > 0 \text{ for } 0 < x < \alpha \text{ and } \varphi(\alpha) = 0 \tag{16}$$

Thus we have the following theorem.

Theorem 2. *If $f(x)$ is continuous for all real x, then for $k > 0$ equation* (4) *has an infinite number of periodic solutions provided there is a c and an $\alpha(c)$ for which* (16) *holds.*

4. Significant special cases of the Roth Problem. In addition the case of the linear $f(x)$ of the original Roth equation (3) there are instances of incomplete investigations of (4) for specific non-linear functions $f(x)$. We will complete these problems to illustrate the methods of Section 3. We will write (4) as the system (13) and then as a Bernoulli equation

$$\frac{dy}{dx} - \frac{ky}{x} = \frac{-f(x) y^{-1}}{x} \tag{17}$$

A. R. Forsyth and W. Jacobsthal [3] (cf., also, E. Kamke [5, p. 578]) determined that

$$x \cos^2(t + c_1) = c_2$$

are solutions of

$$2xx'' - 3x'^2 - 4x^2 = 0 \tag{18}$$

Equation (18) suggests an examination of the equations

$$2xx' \pm 3x'^2 \pm 4x^2 = 0 \tag{19}$$

as typical of what may be done with equations of the form of (4). Except for (18) these equations do not seem to occur in [5], or elsewhere, explicitly.

As for equation (18), the corresponding equation (17) has solutions

$$y^2 = x^2(cx - 4)$$

from which it is clear that the phase portrait consists of a single point ($x=0$, $y=0$) together with parabola-like curves on either side of the $y=x'$ axis. From the portrait, alone, we conclude that each non-trivial solution of (18) has exactly one extreme point and that $|x| \to \infty$ and $|x'| \to \infty$ as $t \to \infty$.

For the equation

(20) $$2xx' - 3x'^2 + 4x^2 = 0$$

the solutions of (17) are

$$y^2 = x^2(4+cx)$$

Since $c=0$ yields the lines $y = \pm 2x$ we discover that

$$x = c_1 \exp[\pm 2(t+c_2)]$$

are solutions of (20). In general the phase portrait has the appearance of a family of folia of Descartes with $y = \pm 2x$ acting as separatrices.

In the case

(21) $$2xx'' + 3x'^2 - 4x^2 = 0$$

the phase space curves are given by

$$y^2 = \frac{4}{5}x^2 + cx^{-3}$$

The lines $y^2 = 4x^2/5$ occur for $c=0$ from which we secure the sub-family of solutions

$$x = c_1 \exp[a(t+c_2)], \quad a^2 = 2/5$$

Otherwise, corresponding to each $c \neq 0$ one secures a curve of three components but, in particular, one component will cut the x-axis and, thus, $x'=0$.

Finally, for

(22) $$2xx'' + 3x'^2 + 4x^2 = 0$$

the phase portrait is given by the family of curves

$$y^2 = -\frac{4x^2}{5} + cx^{-3}$$

from which no solution, except $x \equiv 0$,, is evident in closed form. The solutions of (22) are seen to have a single extreme value of x (an absolute maximum or minimum), the solutions are bounded but $|x'| \to \infty$ as $|t| \to \infty$.

This concludes the analysis of the equations (19). Possibly one further case, $f(x) = kx^4$, should be cited. H. T. Davis [1, p. 223] cconsiders the equation

(23) $$xx'' - \frac{3}{2}x'^2 + kx^4 = 0, \quad k > 0$$

and secures solutions

$$x = \frac{aq}{(t+p)^2+q^2}, \quad a^2 = 2/k$$

An analysis of (23) using equation (17) reveals that for $k>0$ the solutions of (23), in the phase plane, are two families of ovals symmetric with respect to the x-axis each passing through the origin of the phase plane.

For any value of $k \neq 0$ the phase portrait is given by the curves

$$y^2 = x^3 (c - 2kx)$$

If $k<0$, one has the special solutions $y^2 = -2kx^4$ from which it is seen that one has the closed-form solutions of (23),

$$x = \frac{1}{m \pm \sqrt{-2k}\, t} \quad k<0$$

The other solutions for $k<0$ form two familes (corresponding to $c<0$ and $c>0$) of hyperbola-like curves filling the remainder of the phase plane.

In all instances where $f(0)=0$, one has $x \equiv 0$ as a solution of (4), of course.

5. Equations solvable in closed form. In this section we call attention to generalizations of several equations with x'^2 terms to the end of securing a family of solutions in closed form

A. According to E. Kamke [5, p. 590] the equation

$$(x^3+x)x'' - (3x^2-1)x'^2 = 0$$

has solutions

$$x = c, \quad (x^2+1)(c_1 t + c_0) = 1$$

If $f(x)$ is twice differentiable, the problem may be generalized to the equation

$$f'(x) f(x) x'' + (f(x) f''(x) - 2f'^2(x)) x'^2 = 0$$

with solutions

$$x = c, \quad f(x)(c_1 t + c_0) = 1$$

The result is secured by differentiating $c_1 t + c_0 = 1/f(x)$ twice. The problem may be generalized further by starting with $P(t) = 1/f(x)$, where $P(t)$ is a polynomial of degree n with arbitrary coefficients, and then differentiating $n+1$ times.

B. A. R. Forsyth [2] (cf., also, [5, p. 584]) treats

$$t^2(x-1)x'' - 2t^2x'^2 - 2t(x-1)x' - 2x(x-1)^2 = 0$$

by setting $x = 1 + 1/u(t)$ to secure

$$t^2 u'' - 2tu' + 2u = -2 \tag{24}$$

with solutions $u(t) = -1 + c_1 t + c_2 t^2$. Equation (24) is an Euler equation. Thus, one may start with any Euler equation

$$t^2 u'' + ktu' + ru = f(t) \tag{25}$$

and let $u = g(x)$ (in the Forsyth case),

$g(x)=1/(x-1)$, $u'=g'(x)x'$, $u''=g''(x)x'^2+g'(x)x''$. When these are put in (25) one has a tractable equation

$$t^2 g'(x)x''+t^2 g''(x)x'^2+kt\, g'(x)x'+rg(x)=f(x)$$

We have, obviously, assumed $g(x)$ to be twice differentiable.

C. The equation

$$xx''-x'^2-x^2\log x=0$$

is treated by A. R. Forsyth and W. Jacobsthal [3] (cf., also, [5, p. 571]) by setting $u(t)=\log x$ to secure

$$\log x=c_1 e^t+c_2 e^{-t}$$

We observe that if

(26) $$f_1(t)u''+f_2(t)u'+f_3(t)u=0$$

has solutions $u=c_1u_1(t)+c_2u_2(t)$ then

$$\log x=c_1u_1(t)+c_2u_2(t)$$

satisfies

(27) $$f_1(t)xx''-f_1(t)x'^2+f_2(t)xx'+x^2 f_3(t)\log x=0$$

In particular, if (26) is the Euler equation $t^2u''+atu'+bu=0$, then (27) becomes

$$t^2xx''-t^2x'^2+\ +atxx'+bx^2\log x=0$$

and if (26) is the constant coefficient equation $u''+au'+bu=0$, then for (27) one has the tractable equation

$$xx''-x'^2+axx'+bx^2\log x=0$$

In all instances we assume $x(t)>0$

D. To solve

$$xx''+x'^2-x'=0$$

one may let $x'=u$, $x''=u\,du/dx$ to secure the linear equations

$$u=x'=0,\ x\frac{du}{dx}+u-1=0$$

having solutions

$$x=c,\ t=x+c_1\log|x-c_1|+c_2$$

according to H. T. H. Piaggio [8, p. 55], [5, p. 571].

One may make a similar reduction of

$$xx''+x'^2-f(t)x'^{n+1}=0,\ n\geqslant 0$$

to a Bernoulli equation but the result is not attractive

E. To solve the equation

$$2xx''-3x'^2=0$$

E. L. Ince [4], [5, p. 578], divides the equation by xx' to secure the families of solutions

$$x=c,\ x=c_1(t+c_2)^{-2}$$

One may treat the equation

$$xx''+bx'^2=0$$

in the same manner.

F. P. Painlevé [6] observes that

$$x'^2=c\,x(x-1)\exp[-\int f(t)\,dt]$$

is a one-parameter family of first integrals of

$$2x(x-1)x''-(2x-1)x'^2+f(t)x(x-1)x'=0$$

More generally,

$$x'^2=c\,g(x)\exp[-\int f(t)\,dt]$$

is a family of first integrals of

$$2g(x)''-g'(x)x'^2-g(x)f(t)x'=0$$

Trivially, $x\equiv K$ is a solution of the equation.

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