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ON THE SYMMETRIC QUATERNIONIC BANACH ALGEBRAS, I (GELJFAND THEORY)

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Dedicated to Prof. D. S. Mitrinović in honour of his 70-th birthday

1. Introduction

This paper represents an attempt to introduce and investigate the notion of quaternionic Banach algebras, following some classical results from the complex and real theory due to the great masters as I. M. Geljfand, M. A. Naimark, G. E. Shilov and others, expounded for instance in [1] or [4].

We remember that the greater part of the general theory of Banach algebras deals with the algebras over a complex field, due to the strength of the function theoretic methods and the Geljfand representation (which we lose in our algebras).

But despite of the fact that our algebras are only real, so that we cannot apply the methods of complex analysis, we obtain almost all main results of the Geljfand theory of maximal ideals and the fundamental properties of the spectrum of an element. This is rather unexpected, and feasible due to some assumptions ((2.3) and (3.1)) which is natural to introduce.

Throughout the paper the symbols R, C and Q denote the fields of real, complex and quaternionic numbers respectively (R embedded in C, C embedded in Q). The quaternionic units are $e_1=1$, $e_2=i$, $e_3=j$, $e_4=k$. If $\lambda\in C$ or $\lambda\in Q$ then $\bar{\lambda}$ is the conjugate number of λ.

2. Quaternionic Banach algebras

Under a quaternionic Banach algebra B (briefly a QB-algebra, or simply a Banach algebra) we mean an arbitrary unital *two-sided* quaternionic Banach space B in which it holds

1° $rx = xr \quad (r \in R,\ x \in B)$,

2° $\alpha(x\beta) = (\alpha x)\beta \quad (\alpha,\ \beta \in Q)$,

3° $\|\alpha x\| = \|x\alpha\| = |\alpha| \cdot \|x\|$,

4° $\|x+y\| \leqslant \|x\| + \|y\|$,

and the multiplication $(x, y) \to xy$ (which is associative and distributive) possesses the following properties:

5° $\alpha x = (\alpha e)x$ and $x\alpha = x(\alpha e)$' where e is the unit,

6° $\alpha e = e\alpha \quad (\alpha \in Q)$,

7° $\|xy\| \leqslant \|x\| \cdot \|y\|$, and $\|e\| = 1$,

8° $\|x\alpha y\| = |\alpha| \cdot \|xy\|$.

Then immediately holds:

9° $\alpha(xy) = (\alpha x)y$,

10° $(x\alpha)y = x(\alpha y)$,

11° $x(y\alpha) = (xy)\alpha$;

the last three relations are obviously equal if $\alpha \in R$.

Putting

$$B_0 = \{x \in B \mid e_p x = xe_p,\ p = 2, 3, 4\},$$
$$B(i) = \{x \in B \mid e_2 x = xe_2\},$$

we get that $Re(x) := \frac{1}{4} \sum_{p=1}^{4} \bar{e}_p x e_p \in B_0$ for any $x \in B$.

Since $x \equiv \sum_{p=1}^{4} Re(\bar{e}_p x) e_p$ we obtain the direct sums:

(2.1) $$B = \bigoplus_p (e_p B_0) = B_0 \oplus (iB_0) \oplus (jB_0) \oplus (kB_0),$$

(2.2) $$B = B(i) \oplus (jB(i)).$$

It is easily observed that B_0 and $B(i)$ are *real and complex Banach subalgebras of B* respectively.

We remark that such an algebra B is always a real Banach algebra but never *a complex Banach algebra.* The reason is that we lose the important relation $z(xy) = x(zy)$ $(z \in C)$ which enables the application of complex function nethods*).

We introduce the following important assumption:

*) In fact the relation $z(zy) = x(zy)$ holds for every $y \in B$ iff $zx = xz$ that is $x \in B(i)$.

(2.3) *The set* B_0 *is a commutative real subalgebra of* B, or equivalently,

(2.4) *The set* $B(i)$ *is a commutative complex subalgebra of* B,

which we employ throughout the paper.

It immediately follows that the center $Z(B)=\{x \mid xy=yx \text{ for } \forall y\in B\}$ coincides with B_0.

Definition 1. *A mapping* $x\to x^*$ $(x\in B)$ *of a quaternionic Banach algebra* B *is said to be an involution if it possesses the following properties*:

(i) $(x+y)^*=x^*+y^*$,

(ii) $(xy)^*=y^*x^*$,

(iii) $x^{**}=x$,

(iv) $(\alpha x\beta)^*=\bar{\beta}x^*\bar{\alpha}$. □

If $x=\sum x_p e_p\in B$ $(x_1,\ldots,x_4\in B_0)$ let us define the mapping $x\to\bar{x}$ by

$$\bar{x}=\sum_p x_p\bar{e}_p=\sum_p x_p\varepsilon_p e_p,\qquad \text{where}\quad \varepsilon_p=\begin{cases}1, & p=1\\ -1, & p>1.\end{cases}$$

Proposition 1. *The mapping* $x\to\bar{x}$ *is an involution of the algebra* B. □

We call that involution secondary.

Then $\bar{e}=e$, $\bar{x}=x$ iff $x\in B_0$, $\overline{x^{-1}}=\bar{x}^{-1}$ etc.

As a particular case of real Banach algebras we have that every $x\in B$ for which $\|e-x\|<1$ is invertible, and the inverse element

$$x^{-1}=(e-(e-x))^{-1}=\sum_{p=0}^{\infty}(e-x)^p$$

(an absolutely convergent series).

The classical example of such algebras is the set $B=C(X)$ of *all continuous quaternionic valued functions defined on a bicompact Hausdorff space X.*

We think it is quite difficult to provide at least one example of such an algebra in the set of left linear operators $B(H)$ of a two-sided Banach space H. It is especially difficult to satisfy the norm-condition 8° which is very restrictive.

3. Ideals of QB-algebras

3.1. For any subset $E\subseteq B$ let us denote by ad (E) the adherence of E and $\bar{E}=\{\bar{x}\mid x\in E\}$.

We call a set $E\subseteq B$ *symmetric* if $\bar{E}=E$.

We easily see that an $E\subseteq B$ is a left subspace of B iff $\bar{E}$ is its right subspace.

For any left subspace E of B we write:

$$R(E)=\{Re(x)\mid x\in E\},$$
$$H(E)=\sum_p R(E)\,e_p=\bigoplus_p (e_p\,R(E)).$$

Then $R(E)$ is a real subspace of B_0 (not necessarily contained in E), and $H(E)$ is a minimal two-sided subspace of B containing E. Thus $E=H(E)$ iff E is a two-sided subspace, and then ad $(E)=\sum_p e_p$ ad (E).

Proposition 2. *A left subspace E is a two-sided subspace iff it satisfies one of the following equivalent conditions:*

(i) $H(E)=E$;

(ii) $R(E)\subseteq E$;

(iii) $\bar{E}=E$, *i.e. E is a symmetric set;*

(iv) $Ee_p\subseteq E \quad (p=2, 3, 4)$. □

For arbitrary subsets E, F of B, the denotations $E+F$, EF and $\alpha E \beta$ have the usual meaning.

Definition 2. *A subset J of B is its left (or right) ideal if it satisfies*

(i) $J+J\subseteq J$,

(ii) $BJ\subseteq J \quad$ (*or* $JB\subseteq J$). □

It is easy to see that a subset $J\subseteq B$ is a left ideal of B iff $\bar{J}$ is a right ideal of B.

Proposition 3. *If J is a left ideal of B then $R(J)$ is a real ideal of B_0, and $H(J)$ is a two-sided ideal of B (minimal two-sided ideal containing J).* □

Proposition 4. *A left ideal J of B is its two-sided ideal iff it satisfies one of the following equivalent conditions:*

(i) $H(J)=J$,

(ii) $R(J)\subseteq J$,

(iii) $\bar{J}=J$,

(iv) $Je_p\subseteq J \quad (p=2, 3, 4)$. □

In view of the property (iii) we often call two-sided ideals of B *symmetric.*

As in the complex and the real case, *every left (or right) ideal J of B is contained in a maximal left (or right) ideal M, and every maximal left (or maximal right) ideal of B is closed.*

Next we introduce an important assumption which is supposed to be fulfilled throughout the paper:

(3.1) *Every maximal left ideal* M *of* B *is symmetric, that is* $\overline{M}=M$.

This assumption is equivalent to each of the following conditions:

(3.2) *Every maximal right ideal* M *is symmetric*;

(3.3) *Every maximal left* (*or right*) *ideal* M *of* B *is two-sided*;

(3.4) *Every maximal left* (*or right*) *ideal* M *satisfies* $R(M)\neq B_0$, *or equivalently* $H(M)\neq B$;

(3.5) *Every maximal ideal* $M(i)$ *of* $B(i)$ *is symmetric, i.e.* $\overline{M(i)}=M(i)$;

(3.6) *Every left maximal ideal* M *of* B *can be split into the form* $M(i)\oplus jM(i)$, *where* $M(i)$ *is a symmetric maximal ideal of* $B(i)$.

This is the reason why we call such quaternionic Banach algebras *symmetric.*

P r o p o s i t i o n 5. *The left maximal, the right maximal and the two-sidet maximal ideals of* B *are the same.*

P r o o f. Since every left maximal ideal M is two-sided, it is obviously a two-sided maximal ideal of B.

Conversely, if a two-sided maximal ideal M is contained in some larger left maximal ideal M', the ideal M' must be two-sided, which is impossible. □

In view of the preceding property, we will speak only about *the maximal ideals in* B.

P r o p o s i t i o n 6. *An element* $x\neq 0$ *is invertible iff it is not contained in any symmetric ideal of* B.

P r o o f. If x invertible then the set $J(x)=B\cdot x$ is a left proper ideal of B, thus $J(x)$ is contained in some maximal two-sided ideal M.

The contrary is obvious. □

T h e o r e m 1. (i) *A Banach algebra* B *without symmetric ideals is divisible, and each element different from zero is invertible.*

(ii) (Geljfand-Mazur theorem). *Every divisible quaternionic Banach algebra* B *is isomorphic to the field* Q.

P r o o f. (ii) Since B is a division ring, the complex subalgebra $B(i)$ is such a ring too, so that $B(i)$ is isomorphic to the field C. Consequently, every element $x\in B(i)$ is of the form $ze\,(z\in C)$, and element $x\in B$ is of the form $\lambda e\,(\lambda\in Q)$, which implies the required isomorphism. □

3.2 (Quotient algebra). Let M be an arbitrary symmetric closed proper ideal of B. Then the quotient algebra B/M is a two-sided quaternionic Banach algebra which satisfes all conditions $1^\circ - 11^\circ$.

Especially the property $\|x\alpha y\|=|\alpha|\cdot\|xy\|\,(x, y\in B)$ implies $\|X\alpha Y\|=$ $=|\alpha|\cdot\|XY\|\,(X, Y\in B/M)$, and we get

$$(B/M)_0=\{x_0+M \mid x_0\in B_0\},$$
$$(B/M)\,(i)=\{x+M \mid x\in B\,(i)\}.$$

As in the complex theory, there is a one-to-one correspondence between left (or right or symmetric) ideals of the algebra B/M and the corresponding ideals of the algebra B containing the ideal M.

Finally we obtain that the properties (2.3) and (3.1) hold in such algebras too.

Theorem 2. *Let M be any maximal ideal of B. Then the quotient algebra B/M is (bilaterally) isomorphic to the field Q.*

The proof is standard. □

Next let $\mathcal{M}(B)$ be the set of all maximal ideals of B.

Then for every $x\in B$ we obtain a function $\hat{x}:\mathcal{M}(B)\to Q$ (the Geljfand transform of x) defined by relation

$$(*)\qquad x-x\,(M)\,e\in M \quad (\hat{x}\,(M)=x\,(M)).$$

Proposition 7. *The functions $\hat{x}$ $(x\in B)$ possess the following properties:*

(i) $(x+y)\,(M)=x\,(M)+y\,(M)$,

(ii) $(xy)\,(M)=x\,(M)\,y\,(M)$;

(iii) $(\alpha\,x\,\beta)\,(M)=\alpha\,x\,(M)\,\beta$;

(iv) $e\,(M)\equiv 1 \quad (\forall\, M)$;

(v) $x\,(M_0)=0 \Leftrightarrow x\in M_0$;

(vi) $|x\,(M)|\leqslant\|x\|$;

(vii) $M_1\neq M_2 \Rightarrow$ *there is an* $x\in B$ *such that* $x\,(M_1)\neq x\,(M_2)$;

(viii) $x\in B_0 \Rightarrow \hat{x}$ *is a real-valued function*;

(ix) $x\in B\,(i) \Rightarrow \hat{x}$ *is a complex C-valued function, and*

$x\in B_0\,e_p \Rightarrow \hat{x}$ *is a* Re_p*-valued function*;

(x) $\overline{x}\,(M)\equiv\overline{x\,(M)}$ $(\forall\, M)$, i.e. $\hat{\overline{x}}\equiv\overline{\hat{x}}$;

(xi) *If* $x\in B\,(i)$ *and* $M=M\,(i)\oplus j\,M\,(i)\in\mathcal{M}(B)$, *then* $x\,(M)=x\,(M\,(i))$ $(M\,(i)\in\mathcal{M}(B\,(i)))$.

Proof. Proofs of (i) — (vii) are standard.

If further $x\in B_0$ then $\alpha\,x=x\,\alpha\,(\forall\,\alpha\in Q)$, thus $\alpha\,x\,(M)=x\,(M)\,\alpha\,(\forall\,\alpha\in Q)$ so that $x\,(M)\in R$ $(\forall\, M\in\mathcal{M}(B))$. Consequently, the relation (ix) is obvious.

For an arbitrary $x=\sum_p x_p e_p \in B$ we obtain $x(M)=\sum_p x_p(M)e_p$. and

$$\bar{x}(M)=(\sum_p x_p \bar{e}_p)=\sum_p x_p(M)\bar{e}_p=\sum_p \overline{x_p(M)e_p}=\overline{x(M)},$$

which proves (x).

Let now $x\in B$, $M=M(i)\oplus jM(i)$, where $M(i)$ is a symmetric maximal ideal of $B(i)$, and $x(M)=z_1+jz_2$ $(z_1, z_2\in C)$; then by definition

$$x-x(M)e=x-z_1e-jz_2e \quad \in M(i)\oplus jM(i).$$

It follows

$$x-z_1e\in M(i), \qquad z_2e\in M(i),$$

which necessarily implies $z_2=0$ and $z_1=x(M(i))$. Thus $x(M)=z_1=x(M(i))$, which proves (xi). □

4. The maximal ideal space

As usual, we provide the set of maximal ideals $\mathcal{M}(B)$ of B by weak*-topology of left dual space B', the weakest topology in which all the functions $\hat{x}\in\hat{B}$ are contionuous.

The sets

$$U_{(x_1,\dots,x_n;\,\varepsilon)}(M_0)=\{M\,|\,|x_p(M)-x_p(M_0)|<\varepsilon, \quad p=1,\dots,n\}$$

form a fundamental system of neighbourhoods (nbhds) of an ideal $M_0\in\mathcal{M}(B)$, and $\mathcal{M}(B)$ becomes the Hausdorff topological space with this topology.

The maximal ideal space $\mathcal{M}(i)=\mathcal{M}(B(i))$ is endowed also by the weak* -topology in which the sets

$$V_{(y_1,\dots,y_n;\,\varepsilon)}(M_0(i))=\{M(i)\,|\,|y_p(M(i))-y_p(M_0(i))|<\varepsilon, \quad p=1,\dots,n\}$$

$(y_1,\dots,y_n\in B(i))$ form a fundamental system of nbhd's of an ideal $M_0(i)\in\mathcal{M}(i)$.

We have a natural mapping $\pi=\pi(i):\mathcal{M}(i)\to\mathcal{M}$ defined by

$$\pi(i)(M(i))=M(i)\oplus jM(i) \quad (M(i)\in\mathcal{M}(i)),$$

which is obviously one-to-one and onto.

Theorem 3. (i) *The mapping* $\pi(i)$ *is a homeomorphism of the space* $\mathcal{M}(i)$ *onto the space* $\mathcal{M}$.

(ii) *The maximal ideal space* $\mathcal{M}=\mathcal{M}(B)$ *is bicompact.*

Proof. We prove that the mappings $\pi(i)$ and $\pi(i)^{-1}$ are continuous. Let $M_0(i)\in\mathcal{M}(i)$ and $\pi(i)(M_0(i))=M_0$.

If $U_{(x_1,\dots,x_n;\,\varepsilon)}(M_0)=U$ is a nbhd of M_0 $(x_1,\dots,x_n\in B)$, and $x_p=y_p+ +jt_p$ $(1\leqslant p\leqslant n;\ y_p, t_p\in B(i))$, it is easily seen that $\pi(i)V\subseteq U$ where $V= =V_{(y_1,\dots,y_n,\,t_1,\dots,t_n;\,\varepsilon/\sqrt{2})}$ (using Proposition 7, (xi)). Hence π (i) is continuous.

Conversely. for an arbitrary nbhd $V=V_{(x_1,\ldots,x_n;\,\varepsilon)}(M_0(i))$ of $M_0(i)$ $(x_1,\ldots,x_n\in B(i))$ we see that $\pi(i)^{-1}(U)\subseteq V$ where $U=U_{(x_1,\ldots,x_n;\,\varepsilon)}(M_0)$, which implies that $\pi(i)^{-1}$ is continuous. Hence, $\pi(i)$ is a homeomorphism.

Since the space $\mathcal{M}(i)$ is bicompact [1] (p. 38), [4] (p. 233), the second statement is obvious. □

Remark. If $B=C(X)$ is the algebra of all continuous quaternionic functionc defined on a bicompact Hausdorff space X, the preceding theorem implies that the space $\mathcal{M}(B)$ (homeomorphic to the space $\mathcal{M}(B(i))$) is homeomorphic to the space X.

Proposition 8. *The mapping $x\to\hat{x}\ (x\in B)$ is a homomorphism of the algebra B onto an algebra B of continuous functions defined on a bicompact Hausdorff space.*

The kernel of this mapping is radical

$$\mathrm{Rad}(B)=\{x\in B\mid x(M)=0,\ \forall M\in\mathcal{M}\}. \ \square$$

5. The algebra $\hat{B}$ of continuous functions $\hat{x}$

The set $\hat{B}=\{\hat{x}\mid x\in B\}$ (of equivalence classes) forms a Banach algebra with the norm $\|x\|_B=\sup\limits_{M}|x(M)|$.

All properties of the norm $(1°-8°)$ and moreover the property (2.3) are easily verified.

Theorem 4. *Every continuous quaternionic function on the space $\mathcal{M}$ is the limit of a uniform convergent sequence of functions from the algebra $\hat{B}$, that is*

$$C(\mathcal{M})=ad(\hat{B}).$$

Proof. If $f\in C(\mathcal{M})$ then $f=g+jh$, where g, h are continuous complex valued functions, i.e. $g,h\in C_i(\mathcal{M})$.

Then for every $M=M(i)\oplus jM(i)=\pi(M(i))\in\mathcal{M}$ it holds;

$$f(M)=(h\pi)(M(i))+j(h\pi)(M(i)).$$

Since the mapping $\pi:\mathcal{M}(i)\to\mathcal{M}$ is a homeomorphism between $\mathcal{M}(i)$ and $\mathcal{M}$, the mappings $g\pi$, $h\pi$ are continuous complex valued functions, thus $g\pi,\ h\pi\in C_i(\mathcal{M}(i))$.

But since the algebra $B(i)^\wedge$ is „symmetric" ([1], p. 53), the function $g\pi$ is the limit of a uniform convergent sequence $\hat{x}_n\in B(i)^\wedge$ and $h\pi$ is the limit of such a sequence $\hat{y}_n\in B(i)^\wedge$.

Using the relations $x_n(M(i)) = x_n(M)$, $y_n(M(i)) = y_n(M)$ (Proposition 7, (xi)), we obtain

$$g(M) = \text{lim. unif. } x_n(M) \quad (M \in \mathcal{M}).$$

$$h(M) = \text{lim. unif. } y_n(M),$$

thus $f(M) = \text{lim. unif. } v(M)$ $(M \in \mathcal{M})$ where $v_n = x_n + j y_n \in B$. □

Proposition 10. *If* $\|x^2\| = \|x\|^2$ *for every* $x \in B$, *then the algebra* B *is topologically isomorphic to the algebra* $C(\mathcal{M})$.*)

Proof. As an immediate consequence of the Corollary of the Theorem 2 ([1] p. 57) we obtain

$$\|x\|/2 \leqslant \|\hat{x}\| \leqslant 2\|x\| \quad (x \in B).$$

Hence the mapping $\Delta : B \to \hat{B}$ defined by $\Delta(x) = \hat{x}$ is a topological isomorphism of B onto $\hat{B}$.

But then the uniform convergence of functions $\hat{x}_n \in \hat{B}$ implies the norm convergence of the sequence $x_n \in B$, so that $\hat{B}$ is linearly isomorphic to the algebra $C(\mathcal{M})$ (Theorem 4).

It completes the proof. □

6. Spectrum of an element

The spectrum of an element $x \in B$ can be defined at least in two different ways. Namely, since we can construct the complexification B_C of any quaternionic Banach algebra B, we have the possibility to say that the spectrum of an element x B is its complex spectrum in the algebra B_C.

But we adopt another definition which seems to be more advantageous.

Definition 3. *The spectrum* $\sigma(x)$ *of an element* $x \in B \setminus 0$ *is the set* $\{\lambda \in Q \mid x - \lambda e$ *is not invertible*$\}$. □

This spectrum we call *quaternionic*.

Proposition 11. (i) *An element* $x \neq 0$ *is invertible iff* $x(M) \neq 0$ $(\forall M \in \mathcal{M})$;

(ii) *The spectrum* $\sigma(x)$ $(x \in B)$ *is a non-empty set which coincides to the set* $G(\hat{x}) = \{x(M) \mid M \in \mathcal{M}\}$.

Proof. The proof of (i) is standard. If there is not any maximal ideal in B, we obtain that B is isomorphic with Q, but then for any $x = \lambda e \in B$ we get $\sigma(x) = \{\lambda\}$.

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*) We cannot prove that this isomorphism is isometric!

Next (ii) easily follows from (i). □

For an arbitrary $x \in B$ we put $d(x) := \lim \sqrt[n]{\| x^n \|}$ *).

Theorem 5. (i) *Spectrum* $\sigma(x)$ *of an element* $x \in B$ *is a compact subset in the field* Q *lying in the sphere* $|\lambda| \leqslant d(x)$.

(ii) *For any quaternionic rational function* $f(\lambda) = g(\lambda) h(\lambda)^{-1}$ $(\lambda \in Q)$ *with no poles on* $\sigma(x)$ *the spectral mapping theorem holds*:

$$\sigma(f(x)) = f(\sigma(x)).$$

Proof. (i) The resolvent set $\sigma(x) = Q \setminus \sigma(x)$ is open, for if $\lambda_0 \in \rho(x)$, $|\lambda - \lambda_0| < m / \| (x - \lambda_0 e)^{-1} \| = m / \| y_0^{-1} \|$ $(m < 1)$ where $y_0 = x - \lambda_0 e$, then the series $\sum_{p=0}^{\infty} [y_0^{-1} (\lambda - \lambda_0)]^p y_0^{-1}$ is absolutely convergent, so that

$$x - \lambda e = x - \lambda_0 e - (\lambda - \lambda_0) e = (x - \lambda_0 e)(e - y_0^{-1} (\lambda - \lambda_0))$$

is invertible.

Let us prove that if $|\lambda| > d(x)$ the series $-\sum_{p=0}^{\infty} (\lambda^{-1} x)^p \lambda^{-1}$ is absolutely convergent.

By virtue of the norm-assumption 8° we have:

$$\| (\lambda^{-1} x)^p \| = |\lambda|^{-p} \| x^p \| = \| x^p \| \cdot |\lambda|^{-p} \quad (p \in N)$$

so that the radius of convergence of this power series is $d(x)$. Hence $x - \lambda e = -\lambda (e - \lambda^{-1} x)$ is invertible if $|\lambda| > d(x)$, thus $\sigma(x) \subseteq \{\lambda : |\lambda| < d(x)\}$.

Remark. The relation $r_\sigma(x) = d(x)$ holds for every $x \in B(i)$.

We do not know whether $r_\sigma(x) = d(x)$ *holds for every* $x \in B$.

Besides we mention that the estimate $r_\sigma(x) \leqslant d(x)$ is essentially based on the norm-assumption 8°; without this assumption we only have: $r_\sigma(x) \leqslant \| x \|$. *We do not know any better estimate in this general case.*

(ii) By virtue of Proposition 11 (ii), for any quaternionic polynomial

$$(*) \qquad P(\lambda) = P_K(\lambda) = \sum_{m=0}^{K} \sum_{k_1, \ldots, k_{m+1}=1}^{4} a_{k_1 \ldots k_{m+1}} e_{k_1} \lambda \cdots e_{k_m} \lambda e_{k_{m+1}}$$

$(a_{k_1 \ldots k_{m+1}} \in R)$, in the canonical form $(*)$, we easily have the relation $P(x)(M) = P(x(M))$ $(M \in \mathcal{M})$, thus $\sigma(P(x)) = P(\sigma(x))$ (the spectral mapping theorem for polynomials).

Let next $f(\lambda) = g(\lambda) h(\lambda)^{-1}$ be any quaternionic rational function such that the set of poles $S(f) = \{\lambda_0 \in Q \mid h(\lambda_0) = 0\} \subseteq \rho(x)$.

*) The existence of $d(x)$ stems from the general property in real Banach algebras.

Since $0 \notin h(\sigma(x)) = \sigma(h(x))$ we have $h(x)^{-1}$ there exists, so that $f(x) = g(x)\, h(x)^{-1}$ exists too.

But since $y^{-1}(M) = y(M)^{-1}$ (if $y \in B$ is invertible) we obtain

$$f(x)(M) = g(x)\, h(x)^{-1}(M) = g(x)(M)\, h(x)^{-1}(M) = \\ = g(x(M))\, h(x(M))^{-1},$$

wherefrom $\sigma(f(x)) = f(\sigma(x))$. □

From this spectral mapping theorem we have particularly:

$\sigma(\alpha x \beta) = \alpha \sigma(x) \beta$ $(\alpha, \beta \in Q)$, $\lambda(x^{-1}) = \sigma(x)^{-1}$ (if x is invertible),

$\sigma(x^n) = \{\lambda^n \mid \lambda \in \sigma(x)\}$ $\sigma(x + \lambda e) = \sigma(x) + \lambda$, etc.

Proposition 12. (i) *If* $x \in B_0 e_p$ *then* $\sigma(x) \subseteq R e_p$ $(p = 1, 2, 3, 4)$;

(ii) *If* $x \in B(i)$ *then* $\sigma(x) \subseteq C$;

(iii) *For every* $x \in B$, *it holds* $\sigma(\bar{x}) = \overline{\sigma(x)}$.

— The proofs are immediate consequences of Proposition 11. □

We point out the following estimate of the spectral radius.

Proposition 13. *If* $x = \sum_p x_p e_p \in \sum_p B_0 e_p$ *then*

$$d(x_p) \leqslant r_\sigma(x) \leqslant \min\left\{d(x), \sqrt{\sum_p d^2(x_p)}\right\} \qquad (p = 1, 2, 3, 4).$$

Proof. From

$$|x(M)|^2 = \sum_p |x_p(M)|^2 = \sum_p |x_p(M(i))|^2 \quad (M \in \mathcal{M}),$$

using the complex spectral radius formula $r_\sigma(v) = d(v)$ $(v \in B(i))$, we immediately obtain $d(x_p) \leqslant r_\sigma(x)$.

But since, in view of the Theorem 5, $r_\sigma(x) \leqslant d(x)$, and obviously $r_\sigma(x)^2 \leqslant \sum_p d^2(x_p)$, the proof is complete. □

Remark. We want to give a direct deduction of the estimates $d(x_p) \leqslant d(x)$ $(p = 1, 2, 3, 4)$.

Since

$$x_p = \operatorname{Re}(\bar{e}_p x) = \frac{1}{4} \sum_{m=1}^{4} \bar{e}_m e_p x \bar{e}_m,$$

using norm-property 8° we obtain

$$\| x_p^2 \| = \frac{1}{4^2} \left\| \sum_{m, l} \bar{e}_m \bar{e}_p x e_m \bar{e}_l \bar{e}_p x e_l \right\| \leqslant \| x^2 \|,$$

and generally,

$$\| x_p{}^s \| \leqslant \| x^s \| \quad (s=1,\ 2,\ \ldots),$$

which implies $d(x_p) \leqslant d(x)$. □

We do not know whether, in general, there is some estimate of the form

$$(*) \qquad d(x)^2 \leqslant c \sum_p d(x_p)^2 \quad (c>0),$$

or similarly, whether in the general case $d(x_p)=0$ $(p=1, 2, 3, 4)$ *necessarily implies* $d(x)=0$.

We concludes this point with an expected property of the radical.

Proposition 14. (i) *Radical* $\mathrm{Rad}(B)$ *of an algebra* B *is a symmetric ideal which coincides with the intersection of all maximal ideals* $M \in \mathcal{M}(B)$. *It holds*

$$\mathrm{Rad}(B) = \mathrm{Rad}(B(i)) \oplus j\, \mathrm{Rad}(B(i)) = $$
$$= \bigoplus_p \mathrm{Rad}(B_0)\, e_p.$$

(ii) *Element* $x = \sum_p x_p e_p \in \mathrm{Rad}(B)$ *iff* $\sigma(x)=\{0\}$, *or iff* $d(x_1) = \cdots = d(x_4) = 0$, *or if* $d(x)=0$.

Proof. (ii) Obviuosly $x \in \mathrm{Rad}(B)$ iff $\sigma(x)=\{0\}$. In view of (i) it is also equivalent to $x_p \in \mathrm{Rad}(B_0) \subseteq \mathrm{Rad}(B(i))$ which occurs only if $d(x_p)=0$ $(p=1, 2, 3, 4)$.

The last statement is clear in view of the relation $d(x_p) \leqslant d(x)$.

As we have already said, we do not know whether $d(x)$ must be 0 if $x \in \mathrm{Rad}(B)$. □

7. Generators and joint spectrum

A Banach algebra B is said to be finitely generated (by elements $x^{(1)}, \ldots, x^{(n)}$) if every element $x \in B$ can be approximated by a sequence of quaternionic polynomials in $x^{(1)}, \ldots, x^{(n)}$ (generators of B) *).

Theorem 6. *Let a Banach algebra* B *be generated by elements* $x^{(1)}, \ldots, x^{(n)}$. *Then its maximal ideal space* $\mathcal{M}(B)$ *is homeomorphic to a compact subset of the space* Q^n.

Conversely, every compact subset $F \subseteq Q^n$ *is the maximal ideal space of a Banach algebra* B *with* n *generators.*

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*) We take a *polynomial* in $\lambda^1, \ldots, \lambda^n \in Q$ to mean an arbitrary finite sum of products of variables $\lambda^1, \ldots, \lambda^n$ and constants from Q.

Proof. Denoting by

$$F=\{(x^{(1)}(M), \ldots, x^{(n)}(M)) \mid M\in\mathcal{M}\} \subseteq Q^n$$

the joint spectrum of $x^{(1)}, \ldots, x^{(n)}$, and using the standard arguments ([1] pp. 42—46), we obtain that $\mathcal{M}=\mathcal{M}(B)$ is homeomorphic to the subset F, which is compact.

Next we give some comments related to the real and quaternionic polynomial convexity (in R^n and Q^n) which are necessary for the completion of the proof.

A compact set $F\subseteq R^n$ (respectively $F\subseteq Q^n$) is said to be *real (quaternionic) polynomially convex* if for every $\xi=(\xi^1, \ldots, \xi^n)\in R^n\setminus F$ (respectively $\xi\in Q^n\setminus F$) there is a real (quaternionic) polynomial $P(\lambda)=P(\lambda^1, \ldots, \lambda^n)$ such that $P(\xi)=1$ and $|P(\lambda)|<1$ $(\lambda\in F)$.

We assert: *Every compact subset $F\subseteq R^n$ $(F\subseteq Q^n)$ is polynomially convex*, and we prove this.

Let first $F\subseteq R^n$ and $\xi\notin F$, and put:

$$a_p=\max_{\lambda=(\lambda^q)\in F} (\lambda^p-\xi^p)^2 \quad (p=1, \ldots, n),$$

$$b_p=\begin{cases} 1/2\, na_p, & \text{if } a_p=0 \\ 0, & \text{if } a_p=0 \end{cases};$$

define next $P(\lambda)=1-\sum_{p=1}^{n} b_p(\lambda^p-\xi^p)^2$.

Then at least one among $b_1, \ldots, b_n$ is different from zero, and we have $P(\xi)=1$, $P(\lambda)\geqslant 1/2>0$ $(\lambda\in F)$ because $b_p(\lambda^p-\xi^p)^2\leqslant b_p a_p\leqslant 1/2n$ $(p=1, \ldots, n)$.

Hence $\max_{\lambda\in F}|P(\lambda)|=\theta<1$ (F is compact), thus $1/2\leqslant\theta<1$. □

Further, identifying Q^n with R^{4n} (as real vector space), we easily see that an $F\subseteq Q^n$ is quaternionic polynomially convex if and only if it is really polynomially convex (in R^{4n}).

To verify this, it is sufficient to remark that every real polynomial $P(\lambda_1^1, \ldots, \lambda_4^n)$ in R^{4n} is a special quaternionic polynomial, for every real component $\lambda_p^\nu=Re(\bar{e}_p\lambda^\nu)=\frac{1}{4}\sum_m \bar{e}_m(\bar{e}_p\lambda^\nu)e_m$ of λ^ν $(1\leqslant\nu\leqslant n,\ 1\leqslant p\leqslant 4)$ is a special quaternionic polynomial in $\lambda^1, \ldots, \lambda^n$.

We now complete the proof of the theorem.

Using the spectral mapping theorem for joint spectrum ane polynomial functions of n variables (which can be proved exactly as Theorem 5), we can apply the same arguments as in [1] (p. 45) or [2] (p. 101). □

8. Shilov boundary.

Let B be a quaternionic Banach algebra, $\mathcal{M}(B)$ being its maximal ideal space.

If $\hat{x}\in\hat{B}$ and $F\subseteq\mathcal{M}(B)$ is any closed subset in the maximal ideal space, let us denote the semi-norm $\|\hat{x}\|_F=\sup_{M\in F}|x(M)|$. The set F is said to be *maximizing* (for $\hat{B}$) if $\|\hat{x}\|_F=\|\hat{x}\|$.

Then *the Shilov boundary* $\check{S}(B)$ *of* $\hat{B}$ *is the intersection of all maximizing sets of* $\hat{B}$.

Proposition 15. *There exists a unique Shilov boundary* $\check{S}(B)$ *in the maximal ideal* $\mathcal{M}(B)$.

— The proof is exactly the same as in [1] (pp. 75—77), or [4] (pp. 250—251). □

Proposition 16. *A maximal ideal* $M_0\in\mathcal{M}(B)$ *belongs to the set* $\check{S}(B)$ *if for every nbhd* $V(M_0)$ *of* M_0 *there is a function* $\hat{x}\in\hat{B}$ *such that* $\max_{M\in V(M_0)}|x(M)|=$ $=\|\hat{x}\|$ *and* $|x(M)|<\|\hat{x}\|$ $(M)\in\mathcal{M}\setminus V)$. □

Theorem 7. *The Shilov boundary* $\check{S}(B)$ *coincides with the entire maximal ideal space* $\mathcal{M}(B)$.

Proof. This proprerty is a direct consequence of the corresponding theorem in complex symmetric algebras [4] (p. 258). Namely, as in the quoted reference, for any $M_0\in\mathcal{M}$ and any open nbhd V of M_0, there is a continuous real function f $(0\leqslant f(M)\leqslant 1, \forall M)$ defined on $\mathcal{M}(B)$ such that $f(M_0)=1$, $f(\mathcal{M}\setminus V)=0$.

But then, by virtue of Proposition 16, there is an $x\in B$ such that $|f(M)-x(M)|<1/3$ $(\forall M\in\mathcal{M}(B))$, thus $\|\hat{x}\|=\|\hat{x}\|_V$, and $|x(M)|<\|\hat{x}\|$ $(M\in\mathcal{M}\setminus V)$; therefore $M_0\in\check{S}(B)$, q. e. d. □

9. Extensions of maximal ideals

Theorem 8. *Let B and B' be two Banach algebras with the same unit e, and let B be imbedded in B'. Then every maximal ideal of B can be extended to a maximal ideal of B'.*

For every $x\in B$ it holds:

$$(*)\qquad \frac{1}{2}r_\sigma'(x)\leqslant r_\sigma(x)\leqslant r_\sigma'(x).$$

Proof. We use the classical results from [1] (p. 82, Th. 1) or [4] (p. 252), and the preceding theorem.

Let us consider any maximal ideal $M = M(i) \oplus jM(i)$ of B.

Since the complex Banach subalgebra $B(i)$ is *symmetric*, by Theorem 3 of [4] (p. 258) we have that $\check{S}(B(i)) = \mathcal{M}(i)$, so the quoted theorem implies that there exists an extension $M(i)'$ of $M(i)$ in the algebra B'. But $M(i)'$ is a symmetric maximal ideal, so that $M' = M(i)' \oplus jM(i)'$ is a required extension of M from the space $\mathcal{M}(B')$.

Next, since $|x(M)|^2 = |y(M(i))|^2 + |t(M(i))|^2$ $(x = y + jt \in B(i) \oplus jB(i))$, the relation $\sup_{M(i)} |v(M(i))| = \sup_{M(i)'} |v(M(i)')|$ $(v \in B(i))$ imples the above inequalities.

We think it is interesting to see *when the equality* $r_\sigma(x) = r_\sigma(x)'$ *does not hold*. It holds at least if $x \in B(i)$, $x \in qB(i)$ $(q \in Q)$ etc. □

10. Decomposition of a Banach algebra into a direct sum of ideals

Proposition 17. *If* $B = J_1 \oplus J_2$ *is a direct sum of symmetric ideals* J_1, J_2 *and* e_1, e_2 *are the components of unit* e, *then* e_p *is the unit of the closed subalgebra* J_p $(p = 1, 2)$.

Vice versa, let there be an $e_1 \in B_0$ *such that* $e_1{}^2 = e_1$ $(e_1 \neq 0, e)$. *Then* B *is the direct sum of symmetric ideals* $J_p = B \cdot e_p$ $(p = 1, 2)$ *generated by* e_1 *and* $e_2 = e - e_1$.

— Proof is very similar to the ordinary one ([1], p. 101). We only observe that $e = e_1 + e_2 \in J_1 \oplus J_2$ implies $e_1, e_2 \in B_0$. □

Theorem 9. *If* $B = J_1 \oplus J_2$ *is the direct sum of two symmetric ideals* J_1 *and* J_2, *then the space* $\mathcal{M}(B)$ *is the union of two disjoint closed sets* F_1, F_2 *where* $F_p = \mathcal{M}(J_p)$ $(p = 1, 2)$. □

Theorem 10. *Let the space* $\mathcal{M}(B)$ *be the disjoint union of two closed sets* F_1, F_2. *Then* B *is the direct sum* $J_1 \oplus J_2$ *of two unital subalgebras* J_1, J_2 *and* $F_p = \mathcal{M}(J_p)$ $(p = 1, 2)$.

Proof. We prove that there is an element $e_1 \in B_0$ such that $e_1{}^2 = e_1$ and $F_1 = \{M \in \mathcal{M}(B) \mid e_1(M) = 1\}$.

If $M = M(i) \oplus jM(i)$ where $M(i) \in \mathcal{M}(i)$, let us put

$$F_p(i) = \{M(i) \mid M = M(i) \oplus jM(i) \in F_p\} \quad (p = 1, 2).$$

Then $\mathcal{M}(B(i)) = F_1(i) \dot{\cup} F_2(i)$.

By virtue of Theorem 2 in [1] (p. 102—103), there exists some $v \in B(i)$ such that

$$v(M(i)) = \begin{cases} 1, & M(i) \in F_1(i) \\ 0, & M(i) \in F_2(i) \end{cases}.$$

Thus for every $M(i) \in F(i)$ it holds $v^2(M(i)) = v(M(i)) \in \{0, 1\}$, so that $v^2 - v \in \operatorname{Rad}(B(i))$.

Denoting next $w = Re(v) = \frac{1}{2}(v + \bar{v})$ we have

$$w(M(i)) = Re\, v(M(i)) = v(i)),$$

thus for every $M(i) \in F(i)$ it holds $w(M(i)) \in \{0, 1\}$, in other words

$$w^2(M(i)) \equiv w(M(i)) \quad (w \in B_0).$$

Let us define next, as in [1] (p. 103):

$$(*) \qquad e_1 = e_1(w) = -\frac{1}{2\pi i} \int\limits_{|z-1|=1/2} (w - ze)^{-1}\, dz.$$

Since the circle $|z-1| = 1/2$ is symmetric with respect to the real axis, we obtain

$$\bar{e}_1 = \overline{e_1(w)} = -\frac{1}{2\pi i} \int\limits_{|z-1|=1/2} (\bar{w} - ze)^{-1}\, dz =$$

$$= e_1 \quad (\text{for } \bar{w} = w).$$

Thus the element e_1 (which is different from 0 and from e) belongs to the real subalgebra B_0, q. e .d.

Moreover, it holds $e_1{}^2 = e_1$ and the Proposition 17 completes the proof. □

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