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ON THE DECOMPOSITION OF THE LINE (TOTAL) GRAPHS WITH RESPECT TO SOME BINARY OPERATIONS

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Abstract: **It is proved that line and total graphs (and their complements too) are nearly always prime, i.e. excluding some exceptions, they could not be decomposed with respect to some binary operations (sum, product and strong product). In other words, the corresponding graph equations have not too many solutions.**

0. Introduction

In this paper we shall consider only finite, undirected graphs without loops or multiple edges. Throughout the paper $\bar{G}$, $L(G)$, $T(G)$, as customary, will denote the complement, the line graph and the total graph of G. Further, $\cup$ (in some papers $\dot{+}$) and ∇ (in [1] it is denoted by $+$) will stand for union and join of graphs. We shall also prefer to use $+$ for the cartesian product, $\times$ for the conjunction and $*$ for the strong product of some graphs. Next, we shall assume that two graphs G and H are equal if they are isomorphic

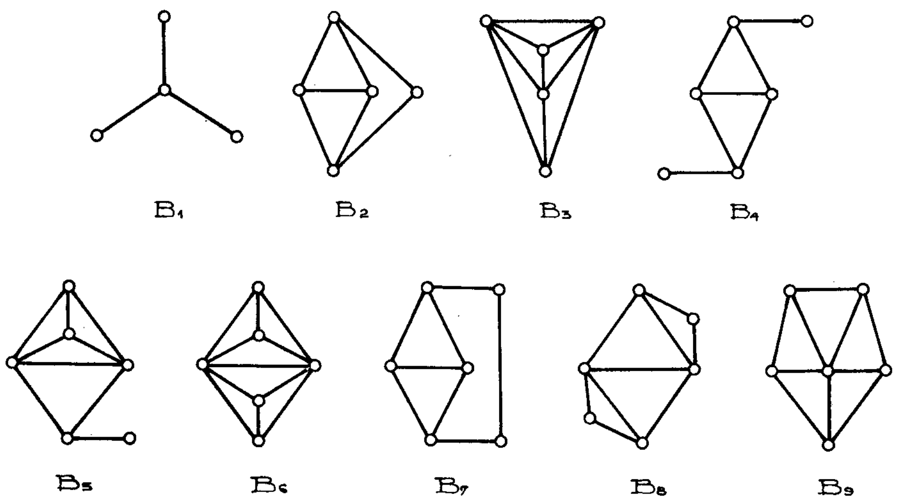

Fig 1.

and this will be denoted by $G=H$. If G is an induced subgraph of H, we shall write $G \subseteq H$. For the rest of notations, see [1]. Especially for the definition of graph equations, see [2] or [3].

For our further purposes we quote some facts about line and total graphs.

Theorem 0.1. G is a line graph if and only if none of the nine graphs from Fig. 1 is an induced subgraph of G.

Theorem 0.2. If G and H are connected graphs such that $L(G)=L(H)$ then $G=H$ unless one of them is K_3 and the other is $K_{1,3}$.

These are well known theorems of L. W. Beineke and H. Whitney, for line graphs. There are also analogous results for total graphs (one can find them in [4], [5], [6]). Here we shall only give some lemmas which can be easily verified. Primarily, we need a concept of v-vertex and e-vertex of a total graph. If $H=T(G)$ then some vertex of H is the v-vertex (e-vertex) if it originates from some vertex (edge) of G. Note, according to [4] if G has no cycles or complete graphs as components, then v-vertices or e-vertices are uniquely determined in H. Now, we give the following six lemmas, first three of which, can also be found in [7].

Lemma 1. In $T(G)$ any e-vertex is adjacent to precisely two v-vertices.

Lemma 2. If in $T(G)$ the e-vertex e is adjacent to v-vertices u and v, then u and v are adjacent.

Lemma 3. If in $T(G)$ the v-vertex v is adjacent to e-vertices e and f, then e and f are adjacent.

Lemma 4. If $K_{1,3} \subseteq T(G)$, then the „central" vertex of $K_{1,3}$ is the v-vertex.

Lemma 5. If in $T(G)$ two v-vertices are adjacent, then there is just one e-vertex adjacent to both of them.

Lemma 6. If in $T(G)$ two e-vertices are adjacent, then there is just one v-vertex adjacent to both of them.

In Section A we shall treat the problems of the decomposition of the line graphs (and their complements too) according to binary operations referred as sum $+$, product $\times$ and strong product $*$, while in Section B the analogous problems will be considered for total graphs. Some interesting remarks on total graphs are given in the Appendix.

1. Section A

In [8] or [9] one can find the following result, which is here restated in the form of graph equation.

Theorem A1. Graph equation $L(G)=G_1+G_2$, for G being connected and G_1, G_2 being nontrivial, has the following solutions: $(G, G_1, G_2)= =(K_{m,n}, K_m, K_n)$.

Of course, the restrictions given for graphs G, G_1, G_2 do not, in essence, change the generality of the above result. In further text we shall use to make some similar restrictions in order to avoid any of uninteresting cases.

Theorem A2. *Graph equation* $\overline{L(G)}=G_1+G_2$, *for* G *having no isolated vertices and for* G_1, G_2 *being nontrivial, has the following solutions*:

$$(G, G_i, G_j)=(C_6, K_2, K_3),\ (K_{3,3}, K_3, K_3),\ (K_{1,mn}, mK_1, nK_1),\ (C_4, 2K_1, K_2),$$

$$(K_4, 3K_1, K_2),\ (2K_{1,2}, K_2, K_2),\ (4K_{1,2}, K_2, C_4),\ (K_{2,4}-2K_2, K_2, K_{1,2})^*,$$

where $i \neq j$, $i, j=1, 2$,

Proof. Suppose first $K_3 \subseteq G_i$. But then G_j must be complete since otherwise $K_1 \cup K_3 \subseteq 2K_1+K_3 \subseteq G_1+G_2$, i.e. $B_1 \subseteq \overline{L(G)}$ (see Fig. 1). In the same way, it follows that G_j cannot have more than 3 vertices and that G_i cannot be a supergraph of K_3 (otherwise $K_1 \cup K_3$ appears again in G_1+G_2). Hence, $(G_i, G_j)=(K_3, K_2), (K_3, K_3)$ and we get G easily.

Now let $K_3 \nsubseteq G_1, G_2$. Consequently $K_3 \nsubseteq G_1+G_2$. Graphs $H=\overline{L(G)}$ (for some G) having no triangles were described in [10]. For our purpose the following observations are sufficient.

(a) $K_{1,2}$ is not an induced subgraph for both G_1 and G_2. Namely, if $K_{1,2} \subseteq G_1, G_2$, then $K_2 \cup K_{1,2} \subseteq K_{1,2}+K_{1,2} \subseteq G_1+G_2$, i.e. $B_2 \subseteq \overline{L(G)}$ (see Fig. 1).

(b) If G_i has at least one edge, then P_4 or $K_{1,3}$ are not induced subgraphs of G_j. For $P_4 \subseteq G_j$, since $K_2 \cup K_{1,2} \subseteq K_2+P_4 \subseteq G_1+G_2$, we get the same contradiction as with (a). For $K_{1,3} \subseteq G_j$, then $K_2+K_{1,3}$ contains as an induced subgraph a graph obtained from a cycle of length 4 by adding two pendant edges to the same vertex of the cycle. Since the complement of the last graph is B_5 (see Fig. 1) we arrive to a contradiction.

(c) If $K_1 \cup K_2$ is an induced subgraph of G_i, then $G_j=K_1$. Namely, if G_j has at least two vertices, then $K_2 \cup K_{1,2}$ or $K_2 \cup 3K_1$ are induced subgraphs of G_1+G_2. Since the complements of the above two graphs are B_2 and B_3 (see Fig. 1) we arrive to a contradiction.

Now we shall summarize the above observations. Clearly, G_1 and G_2 can be both without edges and then we find G easily. If only G_i is without edges then $\overline{L(G)}=p(G_i)\,G_j$ $(p(G_i) \geqslant 2)$ and again if we want to avoid $B_2 \subseteq \overline{L(G)}$ or $B_3 \subseteq \overline{L(G)}$ it follows $p(G_i) \leqslant 3$ and $G_j=K_2$. Thus, $(G_i, G_j)=(2K_1, K_2)$, $(3K_1, K_2)$ and for both cases G can be easily found. So, let both G_1, G_2 have edges. Due to (c) G_1, G_2 are both connected and since they are as well triangle-free according to (a) say $G_i=K_2$ and from (b) G_j can be equal only to K_2, $K_{1,2}$, C_4. By direct checking we get that in all three possibilities G exists. ⊠

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* $K_{m,n}-pK_2$ is obtained from $K_{m,n}$ by removing p independent edges.

Theorem A3. *Graph equation $L(G)=G_1\times G_2$ for G having no isolated vertices and for G_1, G_2 being connected and nontrivial, has the following solutions*:

$$(G, G_i, G_j)=(2P_{n+2}, P_{n+1}, P_2),\ (2C_{2n+2}, C_{2n+2}, K_2),\ (C_{4n+2}, C_{2n+1}, K_2),$$
$$(K_{3,3}, K_3, K_3), \text{ where } i\neq j,\ i, j=1, 2.$$

Proof. Suppose that maximal vertex degree of G_1 or G_2 is greater than 2. Then, since G_1 and G_2 both have edges, it is easy too see that $K_{1,3}\subseteq G_1\times G_2$, i.e., $K_{1,3}\subseteq L(G)$. The last contradiction gives that G_1 and G_2, due to connectedness, are paths or cycles. For $K_{1,2}\subseteq G_i$ it follows that $G_j=K_2$, since otherwise (if G_j contains $K_{1,2}$ or K_3) $K_{1,3}$ appears again in $G_1\times G_2$. Collecting the above conclusions we arrive at the proof of the theorem. ⊠

The following two theorems are simple for proving and we only quote them.

Theorem A4. *Graph equation $\overline{L(G)}=G_1\times G_2$, for G having no isolated vertices and for G_1, G_2 being nontrivial and not totally disconnected, has the following solutions*:

$$(G, G_1, G_2)=(K_{m,n}, K_m, K_n).$$

Theorem A5. *Graph equation $L(G)=G_1*G_2$, for G being connected and for G_1, G_2 being nontrivial, has the following solutions*:

$$(G. G_1, G_2)=(K_{1,mn}, K_m, K_n).$$

Theorem A6. *Graph equation $\overline{L(G)}=G_1*G_2$, for G having no isolated vertices and for G_1, G_2 being nontrivial, has the following solutions*:

$$(G, G_i, G_j)=(K_{1,mn}, mK_1, nK_1),\ (klK_2, K_k, K_l),\ (2n_1K_2\cup n_2C_4\cup n_3K_4, K_2,$$
$$\overline{n_1K_1\cup n_2K_2\cup n_3K_3}), \textit{ where } i\neq j,\ i, j=1, 2.$$

Proof. If G_1 or G_2 have no edges the same situation as with Theorem A2 appears. So let both G_1 and G_2 have edges. If G_1 and G_2 are both complete we immediately get the corresponding graphs G. Thus, take that at least one of G_1 and G_2 is incomplete. Now, by observing that $H_1\subseteq G_1$ and $H_2\subseteq G_2$ implies $H_1*H_2\subseteq G_1*G_2$ we easily get that G_1 and G_2 have not $K_2\cup K_1$ as induced subgraphs, since otherwise $K_1\cup K_3\subseteq(K_2\cup K_1)*K_2$, i.e. $K_{1,3}\subseteq L(G)$ (note G_1 and G_2 both have edges). Hence, if G_j is incomplete it must contain $K_{1,2}$ as an induced subgraph. But, then G_j cannot contain $K_{1,2}$ or K_3 as induced subgraphs since $K_1\cup K_3$ appears in $K_{1,2}*K_{1,2}$ and $K_3*K_{1,2}$ as an induced subgraph. Thus $G_i=K_2$. For $4K_1\subseteq G_j$ we have $K_2\cup 3K_1\subseteq G_1*G_2$, i.e. $K_5-x\subseteq L(G)$ (contradiction due to Theorem 0.1). Now we have $G_j=n_1K_1\cup n_2K_2\cup n_3K_3$. Since for each value of n_1, n_2, n_3 G exists we have found all the solutions. ⊠

2. Section B

Now we shall prove the analogous theorems for totalgraphs.

Theorem B1. *Graph equation $T(G)=G_1+G_2$, for G_1, G_2 being connected and nontrivial, has no solutions.*

Proof. Since both G_1, G_2 have edges the same holds for G. But then $T(G)$ contains a triangle and hence G_1 or G_2 must contain a triangle. Since G_1 and G_2 both have edges to each triangle in G_1+G_2 there corresponds another triangle such that they together induce a prism K_3+K_2. Let us show that in $T(G)$ there is a triangle without just mentioned property. Observe the triangle euv in $T(G)$ such that e is an e-vertex and u, v are v-vertices. Suppose that x is a vertex of another triangle (which corresponds to the first one) and take that x is adjacent to e and not to u and v. By Lemma 1, x must be an e-vertex. Now, using Lemma 6 we easily get that x must be adjacent to u or v. ⊠

Theorem B 2. *Graph equation* $\overline{T(G)}=G_1+G_2$, *for* G_1, G_2 *being nontrivial, has the following solutons*:

$$(G,\ G_i,\ G_j)=(K_3,\ 3K_1,\ K_2), \text{ where } i\neq j,\ i,\ j=1,\ 2.$$

Proof. According to Theorem C 1 (see the Appendix) $\overline{T(G)}$ is disconnected only if $G=K_3$ or $K_{1,n}$. For $G=K_3$ we easily get G_1 and G_2, while for $G=K_{1,n}$ there are no nontrivial solutions. Hence, we have that $\overline{T}(G)$ is connected, what implies the same for G_1 and G_2. If G_1 and G_2 are triangle-free the same holds for G_1+G_2. In that case due to Theorem C 2 (see the Appendix) G_1+G_2 could be one of the following nine graphs: K_1, $3K_1$, K_2, $3K_2$, $K_{1,3}$, $K_{3,3}$, $P_4\cup K_1$, Möbius ladder on eight vertices and Petersen's graph. By direct checking we confirm ourselfs that in the above cases there are no additional solutions. So let G_i contain a triangle. Now, if we regard the vertices of G_1+G_2 as an ordered pairs then call $V(k,\ x)$ $(k=1,\ 2)$ the set of all vertices of G_1+G_2 having x for the k-th component. If G_j is incomplete, take $V(j,\ x)$ and $V(j,\ y)$ in such a way that x and y are not adjacent in G_j. In this case it is easy to see, by using Lemma 4, that all vertices in $V(j,\ x)\cup \cup V(j,\ y)$ regarded also as vertices of $T(G)$ are v-vertices. Next, let $u\in V(j,y)$ and take some e-vertex e (it must exist in $T(G)$) adjacent to both u and v. By Lema 1 e is adjacent only to u and v among the vertices from $V(j,\ x)\cup \cup V(j,\ y)$. But now in G_1+G_2 we then have that $e\in V(j,\ z)$ $(z\neq x,\ y)$ is adjacent to all vertices in $V(j,\ x)$ expect u. Of course, this is a contradiction since $|V(j,\ x)|=|V(G_i)|\geqslant 3$. Therefore we have that $K_3\subseteq G_i$ implies G_j is complete, and accordingly, two cases are characteristic.

Case a: G_j has more than two vertices. Now $K_3\subseteq G_j$ and G_i is also complete. Then we have to find G such that $\overline{T(G)}=K_m+K_n$ holds. Since $K_m+K_n=\overline{K_m\times K_n}$ we have $T(G)=K_m+K_n$. Due to Theorem B 3 it follows that G does not exist.

Case b: G_j has just two vertices, i.e. $G_j=K_2$. Now $G_i\neq K_3$ (see Case a) and since $K_3\subseteq G_i$ all vertices in $V(j,\ x)$ and $V(j,\ y)$ $(x\neq y)$ except perhaps three ones from each of the above sets are v-vertices (if regarded as vertices of $T(G)$). The last assertion follows directly from Lemma 4. By Lemma 1, since G is not totally disconncted, $p(G_i)-3\leqslant 2$, i.e. $p(G_i)\leqslant 5$. Now it can be easily shown that even in this case there are no solutions. ⊠

Theorem B 3. *Graph equation* $T(G)=G_1\times G_2$, *for* G_1, G_2 *being connected and nontrivial, has no solutions.*

P r o o f. Observing the maximal vertex degrees of $T(G)$ and $G_1 \times G_2$, by using the relations $\Delta(T(G)) = 2\Delta(G)$ and $\Delta(G_1 \times G_2) = \Delta G_1 \, \Delta G_2$, we get

$$\text{(a)} \qquad 2\Delta(G) = \Delta(G_1)\,\Delta(G_2).$$

Similarly, observing the number of vertices in the largest clique of the corresponding graphs we have $c(T(G)) \geqslant \Delta(G) + 1$ and $c(G_1 \times G_2) \leqslant \min(c(G_1), c(G_2)) \leqslant \min(\Delta G_1), \Delta(G_2)) + 1$. Therefore, it follows

$$\text{(b)} \qquad 2\Delta(G) \leqslant \Delta(G_1) + \Delta(G_2).$$

Combining (a) and (b) we get

$$\text{(c)} \qquad (\Delta(G_1) - 1)\,(\Delta(G_2) - 1) \leqslant 1.$$

For $\Delta(G_i) = 1$ $(i = 1, 2)$, we have $G_i = K_2$ (since G_i connected) and now $G_1 \times G_2$ is bichromatic while $T(G)$ is never bichormatic. So, by using (c) we get $\Delta(G_1) = \Delta(G_2) = 2$. Since G_1 and G_2 are connected they can be only paths or cycles and moreover because they cannot be bichromatic they are cycles of odd lenght, i.e. $G_1 = C_{2m+1}$, $G_2 = C_{2n+1}$. Since $C_{2m+1} \times C_{2n+1} = C_{2m+1} + C_{2n+1}$ according to Theorem B 1 there are no solutions. ⊠

T h e o r e m B 4. *Graph equation $\overline{T(G)} = G_1 \times G_2$, for G_1, G_2 being non-trivial, has no solutions.*

P r o o f. Since $\overline{T(G)}$ is disconnected only for $G = K_3$ or $K_{1,n}$ (see Theorem C 1) it immediately follows that G_1 or G_2 cannot be disconnected. As in Theorem B 2, if we regard the vertices of $G_1 \times G_2$ as ordered pairs then call $V(k, x)$ $(k = 1, 2)$ the set of all vertices of $G_1 \times G_2$ having x for the k-th component. Now in $G_1 \times G_2$ all vertices in $V(i, x)$ $(i = 1, 2)$ are mutually non-adjacent. In $\overline{G_1 \times G_2}$ the corresponding vertices are all mutually adjacent. Hence, $\overline{G_1 \times G_2}$ can be partitioned into $p(G_i)$ complete subgraphs with $p(G_j)$ $(j \neq i, j = 1, 2)$ vertices. If $p(G_j) \geqslant 4$ the vertices of the mentioned complete subgraphs regarded also as vertices of $T(G)$ can be according to Lemma 5, only of the following types:

(a) all vertices in one complete subgraph are v-vertices;

(b) all vertices in one complete subgraph except one which is a v-vertex are e-vertices:

(c) all vertices in one complete subgraph are e-vertices.

Assume that in $\overline{G_1 \times G_2}$ there are no complete subgraph of the type (a). Then we immediately get a contradiction since in that case G would have more edges than a complete graph with the same number of vertices. Now suppose that more than one of such complete subgraphs exist and observe just two having for their vertex sets $V(i, x)$ and $V(i, y)$ $(x \neq y)$. Next let an e-vertex e be adjacent to two (adjacent) vertices from $V(i, x)$. By Lemma 1 it cannot be adjacent to any vertex from $V(i, y)$. Hence, in $\overline{G_1 \times G_2}$ there exists a vertex being adjacent to all vertices from $V(i, x)$ but not to all vertices from $V(i, y)$ (note $p(G_j) \geqslant 4$). Therefore, it follows that only one complete subgraph of the type (a) may exist in $\overline{G_1 \times G_2}$. Denote it by C^1 ($V(i, x)$ is its vertex-set) and observe an e-vertex e adjacent to two vertices, say u, v from C^1. Of course, e belongs to some other complete subgraph, say C^2 having $V(i, y)$ as its vertex-set, and due to Lemma 1 C^2 is of the type (c). Hence, in $T(G)$ due to Theorem 0.2 there is a star as a subgraph (not necessarly induced) having $p(G_j)$ edges so that all its vertices are v-vertices and u or v is its "central" vertex. Now since C^1 contains $p(G_j)$ vertices at least one of the vertices of the mentioned star does not belong to C^1. Denote one one such vertex by w and take that it belongs to some complete subgraph C^3 (let the vertices of C^3 be $V(i, z)$, $z \neq x, y$). As follows from above, C^3 must be of the type (b). Now take a vertex f from C^2 which is adjacent to w. Using Lemma 2 it follows that f is adjacent to all e-vertices from C^3. But now in $G_1 \times G_2$ f is not adjacent to all vertices in $V(i, z)$ while e is adjacent to $w \in V(\iota, z)$ and also we have $e, f \in V(i, y)$. The last is possible only if G_1 or G_2 have isolated vertices which contradicts the fact that G_1 and G_2 are connected. If $p(G_i) \leqslant 3$ $(i = 1, 2)$, by direct checking, we get that there are no solutions. ⊠

T h e o r e m B 5. *Graph equation* $T(G) = G_1 * G_2$, *for* G_1, G_2 *being connected and nontrivial, has no solutions.*

P r o o f. The case when G is without edge is excluded by suppositions of the theorem. Hence, $T(G)$ contains a triangle and also $G_1 * G_2$. Now we shall prove that to each triangle of $G_1 * G_2$ there corresponds in the same graph a vertex adjacent to all vertices of the observed triangle. For that purpose denote by (u_1, v_1), (u_2, v_2) (u_3, v_3) the vertices of some triangle in $G_1 * G_2$ where u_1, u_2, u_3 (v_1, v_2, v_3) are the vertices of G_1 (G_2). Then one of the following possibilities must hold:

(a) u_1, u_2, u_3 and v_1, v_2, v_3 form triangles in G_1 and G_2, respectively;

(b) u_1, u_2, u_3 (or v_1, v_2, v_3) form a triangle in G_1 (G_2) and just two vertices among v_1, v_2, v_3 (or u_1, u_2, u_3) say v_1, v_2, (or u_1, u_2) represent the same vertex of G_2 (G_1) while the third one v_3 (or u_3) is adjacent to them;

(c) u_1, u_2, u_3 (or v_1, v_2, v_3) form a triangle in G_1 (G_2) and all v_1, v_2, v_3 (or u_1, u_2, u_3) represent the same vertex in G_2 (G_1);

(d) two vertices among u_1, u_2, u_3, say u_1, u_2 and also two vertices among v_1, v_2, v_3 say v_2, v_3 represent the same vertices in G_1 and G_2 respectively, while the third ones u_3 and v_1 are adjacent to u_1, u_2 and v_2, v_3.

Now we shall find (u, v) adjacent to (u_i, v_i) $(i = 1, 2, 3)$.

Case a: It is sufficient to take $u = u_i$, $v = v_j$ $i \neq j$, $i, j \in \{1, 2, 3\}$.

Case b: It is sufficient to take $u = u_3$ (or $v = v_3$) and $v = v_1 = v_2$ (or $u = u_1 = u_2$).

Case c: It is sufficient to take $u = u_i$ (or $v = v_i$) $i \in \{1, 2, 3\}$ and v (or u) is an arbitrary vertex of G_2 (G_1) adjacent to v_i (or u_i).

Case d: Now take $u = u_3$ and $v = v_1$.

To prove the theorem observe a triangle in $T(G)$ having one e-vertex and two v-vertices. Such a triangle has not the above property since otherwise we get contradictions to Lemmas 1 and 5. ⊠

Theorem B6. *Graph equation* $\overline{T(G)} = G_1 * G_2$, *for* G_1, G_2 *being nontrivial, has the following solutions:*

$(G, G_i, G_j) = (mnk_1, K_m, K_n), (2kK_1 \cup lK_3, K_2, kK_1 \cup lK_3)$, *where* $i \neq j$, $i, j = 1, 2$.

Proof. It is easy to see that both G_1 and G_2 can be complete graphs in which case G is totally disconnected. Hence, suppose first that only G_i is complete and also let it be different from K_2 while G_j is incomplete. If the vertices of $G_1 * G_2$ are regarded as as ordered pairs let, as earlier, $V(k, x)$ $(k = 1, 2)$ denote all vertices having x for the k-th component. Now in $\overline{G_1 * G_2}$ vertices $V(j, x)$ for all x are mutually nonadjacent and if $u \in V(j, x)$ then u is adjacent or nonadjacent to all vertices belonging to $V(j, y)$ for some $y \neq x$. So, each vertex in $\overline{G_1 * G_2}$ is isolated or adjacent to three mutually nonadjacent vertices (since $p(G_i) \geqslant 3$). If the vertices of $\overline{G_1 * G_2}$ are now regarded also as the vertices of $T(G)$ by using Lemma 4 it follows that they are all v-vertices and this is obviously a contradiction. Thus G_1 and G_2 are incomplete or one of them is equal to K_2.

For the beginning let both G_1 and G_2 be different from K_2. If G_i contains a triangle then take $V(j, x)$ and $V(j, y)$ so that x and y are nonadjacent. In $\overline{G_1 * G_2}$ each vertex from $V(j, x)$ is adjacent to each vertex from $V(j, y)$ and since G_i contains a triangle by using Lemma 4 it follows that all vertices in $V(j, x) \cup V(j, y)$ are v-vertices. Since G_i is incomplete we can take in $\overline{G_1 * G_2}$ two adjacent vertices $u, v \in V(j, x)$ and an e-vertex $e \in V(j, z)$ $(z \neq x, y)$ which is adjacent to both u and v. By Lemma 1 e is adjacent only to u and v among vertices belonging to $V(j, x) \cup V(j, y)$. Now in $G_1 * G_2$ e is adjacent to all except two vertices from $V(j, x)$ and also it is adjacent to all vertices from $V(j, y)$. This is impossible by the definition of the operation $*$ and thus G_1 and G_2 are both triangle-free. Assume now that $K_1 \cup K_2 \subseteq G_i$. Since G_j is not complete or totally disconnected (the last follows from Theorem C 1) in G_j there exists a vertex being adjacent and nonadjacent to some other vertices of G_j. Hence, it is possible to find in $\overline{G_1 * G_2}$ three groups of vertices, say $V(j, x)$, $V(j, y)$, $V(j, z)$ $(x, y, z$ are mutually different), so that x and y are nonadjacent and x and z are adjacent. Next, let $u_t = (u, t)$, $v_t = (v, t)$, $w_t = (w, t)$ $(t = x, y, z)$ and according to the above we can choose that u_t, v_t, w_t induce $K_1 \cup K_2$ in $G_1 * G_2$. Now in $\overline{G_1 * G_2}$ we have the following induced subgraph, see Fig 2.

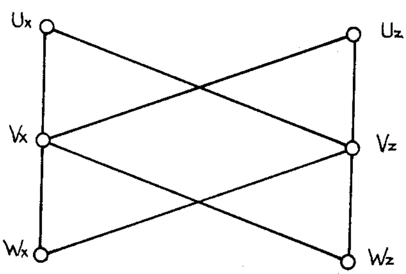

Fig 2.

Now if we regard vertices u_t, v_t, w_t $(t=x, y, z)$ as vertices of $T(G)$ then by Lemma 4 (see Fig. 2) v_x and v_z must be ν-vertices and by Lemma 2 u_x, w_x, u_z, w_z must be also ν-vertices. Hence, all vertices in $V(j, y)$ (see Lemma 1) must be ν-vertices and consequently all vertices in $V(j, x)$ must be ν-vertices. Choose an e-vertex e adjacent to u_x, v_x. By Lemma 1 it is nonadjacent to all other vertices from $V(j, x) \cup V(j, y)$. So in $G_1 * G_2$ e $(e \notin V(j, x) \cup V(j, y))$ is adjacent to all vertices expect two from $V(j, x)$ and also it is adjacent to all vertices from $V(j, y)$. The last is, of course, a contradiction and thus $K_1 \cup K_2$ is not an induced subgraph in G_1 and G_2, Now since $K_1 \cup K_2$ and K_3 are both forbidden in G_1 and G_2 as induced subgraphs and also since neither of G_1 and G_2 is totally disconnected (see Theorem C 1) G_1 and G_2 must be bicomplete graphs. But then it is a matter of routine, by using Lemmas 1 — 6, to verify that the corresponding graph G does not exist. Namely, if $G_1 = K_{m,n}$ and $G_2 = K_{k,l}$ from the above mentioned lemmas for $\max(m, n, k, l) \geqslant 3$, we easily get contradictions; for the rest of possibilities by direct checking we find that G does not exist.

At last it remains the case $G_i = K_2$ and G_j is incomplete. If $K_1 \cup K_2 \subseteq G_j$ then we have the same situation as earlier (see Fig. 2 and corresponding conclusions) i.e. now all vertices in $\overline{G_1 * G_2}$, if regarded as vertices of $T(G)$, are ν-vertices, and this is obviously absurd. Hence, $K_1 \cup K_2$ is forbidden in G_j as an induced subgraph and so $\overline{G}_j = \bigcup_i m_i K_i$. Now $\overline{G_1 * G_2} = \bigcup_i m_i (iK_2)$ and since this graph must also be a total graph all m_i except m_1 and m_3 are equal to 0. ⊠

3. Appendix

Here we will prove two theorems which are used in the above text and which also give some analogous results for complementary total graphs in comparison with the results obtained in [10], for complementary line graphs.

Theorem C 1. *If $H = \overline{T(G)}$ for some G, then H is disconnected if and only if $G = K_3$ or $K_{1,n}$.*

Proof. In fact we have to solve a graph equation $T(G) = G_1 \nabla G_2$. Since G is not totally disconncted we have an e-vertex $e \in V(G_i)$ $(i=1$ or $2)$. But now, since e is adjacent to all vertices from $V(G_j)$ $(j \neq i, j=1, 2)$, by using Lemma 1, we get that $V(G_j)$ has at most two ν-vertices.

Case a: G_j contains just two (adjacent) ν-vertices. Due to Lemma 5 all vertices in G_i except e (if any exists) are ν-vertices. If G_i contains only the vertex e we get $G = K_2$. Otherwise in G_j there must exist at least two e-vertices but due to Lemma 5 G can contain only one ν-vertex. Thus, we get $G = K_3$.

Case b: G_j contains only one ν-vertex, say u. Then G contains a ν-vertex v adjacent to e. If G_j contains at least one e-vertex then, due to Lemma

5, in G_i exist at least two v-vertices and we get the same situation as above. So let u be the only vertex in G_j. Suppose now that in G_i exists a v-vertex w adjacent to v. This is due to Lemma 1 impossible, since otherwise an e-vertex f would be adjacent to u, v and w. If all v-vertices from G_i are mutually nonadjacent we get $G=K_{1,n}$.

Case c: G_j has no v-vertices. Now it is easy to see that except $G=K_2$ there are no other possibilities. ⊠

Corollary. *Graph equation* $T(G)=G_1 \nabla G_2$ *has only the following solutions*:

$$(G,\ G_i,\ G_j)=(K_3,\ 2K_1,\ C_4),\ (K_{1,n},\ K_1,\ K_n \circ K_1),\ \textit{where}\ i \neq j,\ i,\ j=1,\ 2.$$

Here $\circ$ denotes the graph operation corona, see [1].

Theorem C2. *A graph G without triangles is a complement of a total graph if and only if it is equal to one of the following nine graphs*: K_1, $3K_1$, K_2, $3K_2$, $K_{1,3}$, $K_{3,3}$, $P_4 \cup K_1$, *Möbius ladder on eight vevtices and Petersen's graph.*

Proof. Observe that $H_0 \subseteq H$ implies $\overline{T}(H_0) \subseteq \overline{T}(H)$. Now it is easy to see that $3K_1 \subseteq \overline{T}(H_0)$ for each $H_0 \in \{3K_1, K_3 \cup K_1, P_4, K_{1,3}+x, K_4-x, K_5\}$. Thus all graphs from the above set are forbidden as induced subgraphs of H if $\overline{T}(H)$ should be triangle-free. Now it is a matter of routine to find all graphs H (just nine) having the above property. Since their complementary total graphs are triangle-free, the theorem is proved. ⊠

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