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OPERATIONAL RELATIONS RELATED TO A FUNCTION DEFINED BY A GENERALIZED RODRIGUES'FORMULA

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1. Introduction

Following Fuziwara [6] in an attempt to unify classical orthogonal polynomials viz. Laguerre, Hermite, Jacobi etc., Srivastava-Singhal [16] studied a class of polynomial $\{T_n^{(\alpha, \beta)} (x, a, b, c, d, p, r)\}$ defined by a generalized Rodrigues' formula as follows:

$$(1.1) \qquad T_n^{(\alpha, \beta)} (x, a, b, c, d, p, r) = \\ = \frac{(ax+b)^{-\alpha} (cx+d)^{-\beta}}{n!} exp (px^r) \cdot D_x^n \{(ax+b)^{n+\alpha} \cdot (cx+d)^{n+\beta} \cdot e^{-px^r}\}$$

$$\text{where } D_x = \frac{d}{dx}.$$

Simultaneously Singh [13] also studied a generalized polynomial $\{F_{n \cdot \lambda \cdot \mu}^{(c)} [x; \alpha; \beta; h; k; p/r]\}$ defined by the relation

$$(1.2) \qquad F_{n \cdot \lambda \cdot \mu}^{(c)} [x; \alpha, \beta; h, k; p/r] = (x-\lambda)^{-\alpha} \cdot (x+\mu)^{-\beta} \cdot \\ \cdot \exp (px^r) D_x^n \{(x-\lambda)^{kn+\alpha} \cdot (x+\mu)^{hn+\beta} \cdot \exp \cdot (-px^r)\}$$

h, k being non-negative integers, and c, α, μ, λ real numbers.

Then in view of the generalized Rodrigues' formula [9]

$$(1.3) \qquad p_n(x) = \frac{1}{k_n \omega (x)} \cdot D_x^n \{[X(x)]^n \cdot \omega (x)\}$$

and $\Phi_n^{(\lambda)} (x)$ [5] defined by the relation

$$(1.4) \qquad \Phi_n^{(\lambda)} (x) = \frac{k_n}{[X(x)]^\lambda \cdot \omega (x)} \cdot D_x^n \{[X(x)]^{n+\lambda} \cdot \omega (x)\},$$

where $X(x)$ is a polynomial in x of degree$\leqslant 2$.

Srivastava — Panda [15] studied a sequence of functions $\{S_n^{(\alpha, \beta)}[x, a, b, c\ d; \nu, \varepsilon; \omega(x)]\}$ defined by the relation

$$(1.5) \qquad S_n^{(\alpha, \beta)}[x, a, b, c, d; \nu, \varepsilon; \omega(x)] = \frac{(ax+b)^{-\alpha} \cdot (cx+d)^{-\beta}}{n!\, w(x)} \cdot$$

$$\cdot D_x^n \{(ax+b)^{\nu n+\alpha} (cx+d)^{\varepsilon n+\beta} \omega(x)\}$$

where $a, b, c, d, \alpha, \beta, \nu, \varepsilon$ are constants and $\omega(x)$ is independent of n and differentiable an arbitrary number of times.

Going through the above developments and in view of Chak [1], Shrivastava [10], Vijay [18] and Chandel [4], it is of interest to study a sequence $\{S_n^{(\alpha, \beta, k)}[x, a, b, c, d; \nu, \varepsilon; \omega(x)]\}$ defined as

$$(1.6) \qquad S_n^{(\alpha, \beta, k)}[x, a, b, c, d; \nu, \varepsilon; \omega(x)] := \frac{(ax+b)^{-\alpha} \cdot cx+d)^{-\beta}}{n!\, \omega(x)} \cdot$$

$$\theta^n \{(ax+b)^{\nu n+\alpha} \cdot (cx+d)^{\varepsilon n+\beta} \cdot \omega(x)\}.$$

$$\text{where } \theta = x^k \cdot \frac{d}{dx}$$

Evidently following are interesting particular cases:

$$(1.7) \qquad S_n^{(\alpha, \beta, 0)}[x, a, b, c, d; 1, 1; e^{-px^r}] = T_n^{(\alpha, \beta)}[x, a, b, c, d, p, r)$$

$$(1.8) \qquad S_n^{(\alpha, \beta, 0)}[x, 1, -\lambda, 1, \mu; k, h, e^{-px^r}] =$$

$$= \frac{c^n}{n!} F_{n \cdot \lambda \cdot \mu}^{(c)} [x; \alpha, \beta; h, k; p/r]$$

$$(1.9) \qquad S_n^{(\alpha, \beta, 0)}[x, a, b, c, d; \nu, \varepsilon; \omega(x)] = S_n^{(\alpha, \beta)}[x, a, b, c, d; \nu, \varepsilon, \omega(x)]$$

$$(1.10) \qquad S_n^{(\alpha, 0, k)}[x, 1, 0, c, d; 0, 0; e^{-px^r}] = \frac{1}{n!} T_n^{(\alpha, k)}(x, r, p,)$$

$$(1.11) \qquad S_n^{(\alpha, 0, k)}[x, 1, 0, c, d; m, 0, e^{-px^r}] = \frac{1}{n!} F_n^{(r, m)}(x, \alpha, k, p)$$

$$(1.12) \qquad S_n^{(\alpha, 0, k)}[x, 1, 0, c, d; 0, 0; e^{-x}] = \frac{1}{n!} G_{n, k}^{(\alpha)}(x)$$

$$(1.13) \qquad S_n^{(\alpha, 0, 0)}[x, 1, 0, c, d; 0, 0; e^{-px^r}] = \frac{(-1)^n}{n!} H_n^r(x, \alpha, p)$$

Furthermore it is easily verified that

$$(1.14) \qquad S_n^{(\alpha, \beta, k)}[x, a, b, c, d; \nu, \varepsilon; \omega(x)] = [a^{k-1} \cdot b^{\nu+\varepsilon+k-1} \cdot c^{\nu+k-1} \cdot$$

$$\cdot d^{1-k-\varepsilon}] \cdot S_n^{(\alpha, \beta, k)} \left[\frac{bcx}{ad}, a, b, a^2 d, b^2 c; \varepsilon, \nu; \omega(x) \right].$$

Which provides a generalisation of the familiar relationship [17, p. 59]

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x) \tag{1.15}$$

for the classical Jacobi polynomial.

Now we mention below some well known operational relations for the operator $\theta = x^k \dfrac{d}{dx}$ which shall be useful in our study.

They are

$$\theta^n x^\alpha = \alpha^{(k-1,\, n)} \cdot x^{\alpha + (k-1)n} \tag{1.16}$$

where $\alpha^{(k,\, n)} = \alpha \cdot (\alpha + k)(\alpha + 2k) \cdot \cdots \cdot (\alpha + nk - k)$.

$$e^{t\theta} f(x) = f\left(\frac{x}{[1 + (k-1)\, t x^{k-1}]}\,[1/k - 1]\right). \tag{1.17}$$

$$\theta^n (U \cdot V) = \sum_{r=0}^{n} \binom{n}{r} (\theta^{n-r} U) \cdot (\theta^r V). \tag{1.18}$$

$$e^{t\theta}(U \cdot V) = (e^{t\theta} \cdot U)(e^{t\theta} V) \tag{1,19}$$

$$F(\theta)\{x^\alpha g(x)\} = x^\alpha \cdot F(\alpha x^{k-1} + \theta)\, g(x) \tag{1.20}$$

$$F(\theta)\{e^{h(x)} \cdot g(x)\} = e^{h(x)} \cdot F(x^k h'(x) + \theta) \cdot g(x) \tag{1.21}$$

where $h'(x) = \dfrac{dh(x)}{dx}$

2. Operational Formulae

With the help of (1.18), we have

$$\theta^n \{(ax+b)^{\nu n + \alpha} \cdot (cx+d)^{\varepsilon n + \beta} \cdot \omega(x) \cdot Y\} =$$

$$= \sum_{r=0}^{n} \binom{n}{r} \{\theta^{n-r} [(ax+b)^{\nu n + \alpha} (cx+d)^{\varepsilon n + \beta} \cdot \omega(x)]\} \cdot \{\theta^r Y\}.$$

Which by simple manipulations will yield

$$\theta^n \cdot (ax+b)^{\nu n + \alpha} (cx+d)^{\varepsilon n + \beta} \cdot \omega(x) \cdot Y\} = \tag{2.1}$$

$$= n!\, \omega(x) \sum_{r=0}^{n} \frac{(ax+b)^{\nu r + \alpha} \cdot (cx+d)^{\varepsilon r + \beta}}{r!} \cdot$$

$$\cdot S_{n-r}^{(\alpha + \nu r,\, \beta + \varepsilon r,\, k)} [x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)] \cdot \{\theta^r Y\},$$

Next,

$$S_n^{(\alpha,\beta,k)} [x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)] = \frac{(ax + b^{-\alpha} \cdot (cx+d)^{-\beta}}{n!\, \omega(x)} \cdot$$

$$\cdot \theta^n \{(ax+b)^{\nu n + \alpha} \cdot (cx+d)^{\varepsilon n + \beta} \cdot \omega(x)\}.$$

Now,

$$\theta^n \{(ax+b)^{\nu n+\alpha} \cdot (cx+d)^{\varepsilon n+\beta} \cdot \omega(x)\, Y\} =$$

$$= \theta^{n-1} [x^k \{(\nu n+\alpha)(ax+b)^{\nu n+\alpha-1} \cdot a \cdot (cx+d)^{\varepsilon n+\beta} \cdot \omega(x)\, Y +$$

$$+ (ax+b)^{\nu n+\alpha} \cdot (\varepsilon n+\beta)(cx+d)^{\varepsilon n+\beta-1} \cdot c \cdot \omega(x) \cdot Y + (ax+b)^{\nu n+\alpha} \cdot$$

$$\cdot (cx+d)^{\varepsilon n+\beta} \cdot \omega'(x)\, Y + (ax+b)^{\nu n+\alpha} \cdot (cx+d)^{\varepsilon n+\beta} \cdot \omega(x)\, DY].$$

$$= \theta^{n-1} \Big\{(ax+b)^{\nu n+\alpha} \cdot (cx+d)^{\varepsilon n+\beta} \cdot \omega(x) \Big[a(\nu n+\alpha)(ax+b)^{-1} x^k +$$

$$+ c(\varepsilon n+\beta)(cx+d)^{-1} x^k + \frac{x^k \omega'(x)}{\omega(x)} + x^k D\Big] Y\Big\}.$$

$$= \theta^{n-1} \{(ax+b)^{\nu n+\alpha} \cdot (cx+d)^{\varepsilon n+\beta} \cdot \omega(x)\, Y_1\},$$

where

$$Y_1 = \left[\frac{a(\nu n+\alpha) x^k}{ax+b} + \frac{c(\varepsilon n+\beta) x^k}{cx+d} + \frac{x^k \omega'(x)}{\omega(x)} + x^k D\right].$$

Repeating the same procedure once more, we have,

$$= \theta^{n-2} \Big\{(ax+b)^{\nu n+\alpha} \cdot (cx+d)^{\varepsilon n+\beta}\, \omega(x) \Big[a(\nu n+\alpha)(ax+b)^{-1} x^k +$$

$$+ c(\varepsilon n+\beta)(cx+d)^{-1} x^k + \frac{x^k \omega'(x)}{\omega(x)} + x^k D\Big] Y_1\Big\}.$$

On substituting the value of Y_1, we have,

$$= \theta^{n-2} \Big\{(ax+b)^{\nu n+\alpha} \cdot (cx+d)^{\varepsilon n+\beta}\, \omega(x) \Big[a(\nu n+\alpha)(ax+b)^{-1} \cdot$$

$$\cdot x^k + c(\varepsilon n+\beta)(cx+d)^{-1} x^k + \frac{x^k \omega'(x)}{\omega(x)} + \theta\Big]^2 \cdot Y\Big\}$$

Thus, n times repetition will lead us to the operational formula

(2.2)
$$\theta^n \{(ax+b)^{\nu n+\alpha} \cdot (cx+d)^{\varepsilon n+\beta} \cdot \omega(x)\, Y\} =$$

$$= (ax+b)^{\nu n+\alpha} \cdot (cx+d)^{\varepsilon n+\beta} \cdot \omega(x) \cdot \Big\{\frac{a(\nu n+\alpha) x^k}{ax+b} +$$

$$+ \frac{c(\varepsilon n+\beta) x^k}{cx+d} + \frac{x^k \omega'(x)}{\omega(x)} + \theta\Big\}^n \cdot Y$$

This operational formula generalizes the operational formula of Shrivastava [10]

(2.3)
$$\theta^n [x^{\alpha+mn} \cdot e^{-px^r} \cdot f] = x^\alpha \cdot e^{-px^r} \cdot [ax^{k-1} - rpx^{r+k-1} + \theta](x^{mn} \cdot f)$$

Next,

$$\theta^n\{(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot\omega(x)\,Y\}=$$

$$=\theta^{n-1}\left\{(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot\omega(x)\left[\frac{a(\nu n+\alpha)n^k}{ax+b}+\right.\right.$$

$$\left.\left.+\frac{c(\varepsilon n+\beta)x^k}{cx+d}+\frac{x^k\omega'(x)}{\omega(x)}+x^kD\right]Y\right\}.$$

$$=\theta^{n-1}\left\{(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot\omega(x)(x^{-r})\cdot\left[\frac{a(\nu n+\alpha)x^{k+r}}{ax+b}+\right.\right.$$

$$\left.\left.+\frac{c(\varepsilon n+\beta)x^{k+r}}{cx+b}+\frac{x^{k+r}\omega'(x)}{\omega(x)}+x^{k+r}\cdot D\right]Y\right\}.$$

$$=\theta^{n-1}\{(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot\omega(x)(x^{-r})\,Y_1\}$$

where $\left[Y_1=\frac{a(\nu n+\alpha)x^{k+r}}{ax+b}+\frac{c(\varepsilon n+\beta)x^{k+r}}{cx+d}+\frac{x^{k+r}\omega'(x)}{\omega(x)}+x^{k+r}D\right]$

$$=\theta^{n-2}\{x^k[(\nu n+\alpha)\cdot a(ax+b)^{\nu n+\alpha-1}(cx+d)^{\varepsilon n+\beta}x^{-r}\cdot\omega(x)\,Y_1+$$

$$+(\varepsilon n+\beta)c(cx+d)^{\varepsilon n+\beta-1}\cdot(ax+b)^{\nu n+\alpha}\cdot x^{-r}\cdot\omega(x)\,Y_1+$$

$$+(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot(-r)\,x^{-r-1}\cdot\omega(x)\,Y_1+$$

$$+(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot x^{-r}\cdot\omega'(x)Y_1+$$

$$+(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot x^{-r}\cdot\omega'(x)\,DY_1]\}$$

$$=\theta^{n-2}\left\{(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\,\omega(x)(x^{-2r})\cdot\left[\frac{(\nu n+\alpha)}{ax+b}\cdot\right.\right.$$

$$\left.\left.\cdot\frac{ax^{k+r}}{ax+b}+\frac{(\varepsilon n+\beta)c\cdot x^{k+r}}{cx+d}-r\cdot x^{k-1+r}+\frac{x^{k+r}\omega'(x)}{\omega(x)}+x^{k+r}D\right]Y_1\right\}$$

Thus n times repetition will yield

$$\theta^n\{(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot\omega(x)\,Y\}=(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot$$
$$\cdot x^{-nr}\cdot\omega(x)\prod_{j=1}^{n}\left\{\frac{(\nu n+\alpha)ax^{k+r}}{ax+b}+\frac{(\varepsilon n+\beta)cx^{k+r}}{cx+d}\right.$$
$$\left.-(n-j)rx^{k+r-1}+\frac{x^{k+r}\omega'(x)}{\omega(x)}+x^{k+r}D\right\}Y \tag{2.4}$$

Which provides further generalization to the operational formula of Shrivastava [11]

$$D^k[x^{kn+a}\cdot e^{-px^r}\cdot f]=e^{-px^r}\cdot x^{kn+a-n}\cdot\prod_{i=1}^{n}(xD+kn+a-n+i-prx^r)\cdot f. \tag{2.5}$$

Srivastava-Panda [15, *Eq.* 23]

$$(2.6)\qquad \frac{(ax+b)^{-\alpha}\cdot(cx+d)^{-\beta}}{\omega(x)}\cdot D_x^n\{(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot\omega(x)\,Y\}=$$

$$=\left\{\frac{(ax+b)^{\nu}(cx+d)^{\varepsilon}}{x}\right\}^n\cdot\prod_{j=1}^{n}\left[\frac{(\nu n+\alpha)\,ax}{ax+b}+\right.$$

$$\left.+\frac{(\varepsilon n+\beta)\,cx}{cx+d}+\frac{x\,\omega'(x)}{\omega(x)}-(n-j)+xD\right]$$

This formula is put in the opposite operative sense here, and Srivastava-Singhal [16, *Eq.* 27]

$$(2.7)\qquad (ax+b)^{-\alpha}\cdot(cx+d)^{-\beta}\exp\cdot(px^r)\,D_x^n\{(ax+b)^{n+\alpha}\cdot(cx+d)^{n+\beta}\cdot e^{-px^r}\,Y\}=$$

$$=\left\{\frac{(ax+b)(cx+d)}{x}\right\}^n\cdot\prod_{j=1}^{n}\left[xD+\frac{(n+\alpha)\,ax}{ax+b}+\frac{(n+\beta)\,cx}{cx+d}-prx^r-j+1\right]Y$$

Srivastava-Singhal has taken opposite operative sense. Our (2.4) operational formula gives us a set of operational formulae by giving different values to r, when $r=-k$, (2.4) reduces to

$$(2.8)\qquad \theta^n[ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot\omega(x)\,Y]=(ax+b)^{\nu n+\alpha}\cdot(\cdot cx+d)^{\varepsilon n+\beta}\cdot$$

$$\cdot x^{nk}\,\omega(x)\prod_{j=1}^{n}\left\{\frac{(\nu n+\alpha)\,a}{ax+b}+\frac{(\varepsilon n+\beta)\,c}{cx+d}+(n-j)\,kx^{-1}+\frac{\omega'(x)}{\omega(x)}+D\right\}Y.$$

3. Operator Φ

With the help of (1.6) and (1.18), we get

$$(3.1)\qquad \theta^m\cdot S_n^{(\alpha,\,\beta,\,k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]=$$

$$=\frac{1}{n!}\prod_{r=0}^{n}\binom{m}{r}(m-r)!\,(n+r)!\,(ax+b)^{-\nu r}\cdot(cx+d)^{-\varepsilon r}\cdot$$

$$\cdot S_{m-r}^{(-\alpha,\,-\beta,\,k)}\left[x,\ a,\ b,\ c,\ d;\ 0,\ 0;\frac{1}{\omega(x)}\right]\cdot$$

$$\cdot S_{n+r}^{(\alpha-\nu r,\,\beta-\varepsilon r,\,k)}\cdot[x,\ a\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x).$$

which gives when, $m=1$

$$(3.2)\qquad \theta\,S_n^{(\alpha,\,\beta,\,k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]=$$

$$=S_1^{(-\alpha,\,-\beta,\,k)}\left[x,\ a,\ b,\ c,\ d;\ 0,\ 0;\ \frac{1}{\omega(x)}\right]\cdot$$

$$\cdot S_n^{(\alpha,\,\beta,\,k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]+(n+1)\,(ax+b)^{-\nu}\cdot$$

$$\cdot(cx+d)^{-\varepsilon}\,S_{n+1}^{(\alpha-r,\,\beta-\varepsilon,\,k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]$$

which leads to

$$\left(\theta+\frac{\alpha\, ax^k}{ax+b}+\frac{\beta\, cx^k}{cx+d}+\frac{x^k\omega'(x)}{\omega(x)}\right)S_n^{(\alpha,\,\beta,\,k)}[x,\ a,\ b,\ c\ d;$$

$$\nu,\ \varepsilon;\ \omega(x)]=(n+1)\,(ax+b)^{-\nu}(cx+d)^{-\varepsilon}S_{n+1}^{(\alpha-\nu,\,\beta-\varepsilon,\,k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)].$$

Now put

$$\Phi=\theta+\frac{\alpha\, ax^k}{ax+b}+\frac{\beta\, cx^k}{cx+d}+\frac{x^k\omega'(x)}{\omega(x)}.$$

So, that

(3.3) $$\Phi S_n^{(\alpha,\,\beta,\,k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]=(n+1)\,(ax+b)^{-\nu}\cdot$$

$$\cdot(cx+d)^{-\varepsilon}S_{n+1}^{(\alpha-\nu,\,\beta-\varepsilon,\,k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)].$$

Again by iteration, we get,

(3.4) $$\Phi^m S_n^{(\alpha,\,\beta,\,k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]=\frac{(n+m)!}{n!}\cdot$$

$$\cdot(ax+b)^{-m\nu}\cdot(cx+d)^{-m\varepsilon}\cdot S_{n+m}^{(\alpha-m\nu,\,\beta-m\varepsilon,\,k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)].$$

The operator Φ generalises the operators, those given by Gould-Hopper [7]

(3.5) $$\subset s=D_x+\frac{\alpha}{x}-prx^{r-1}$$

and by Singh [14]

(3.6) $$\subset s=xD_x+\alpha-prx^r$$

and is anologous to the operator given by Vijay [18]

(3.7) $$\Phi=x^k hg'+\theta$$

and Shrivastava [12]

(3.8) $$\mathcal{D}=\delta+xhg' \qquad \text{where } \delta=x\tfrac{d}{dx}$$

when, $n=0$ relation (3.4) reduces to

(3.9) $$\Phi^m\cdot 1=m!\,(ax+b)^{-m\nu}\cdot(cx+d)^{-m\varepsilon}\cdot S_m^{(\alpha-m\nu,\,\beta-m\varepsilon,\,k)}[x,$$

$$a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)].$$

This is an operational formula which happens to give many special functions in particular cases. For example Chandel [4]

(3.10) $$[x^k hg'+\theta]^n\cdot 1=G_n(h,\ g,\ k)$$

and Shrivastava [12]

(3.11) $$[\delta+xhg']^n\cdot 1=G_n(h,\ g)$$

Next, it can be easily verified that

(3.12) $$\Phi^n(U\cdot V)=\sum_{i=0}^{n}\binom{n}{i}(\theta^i\cdot U)(\Phi^{n-i}\cdot V).$$

Put $U=f$ and $V=1$, (3.12) yields.

(3.13) $$\Phi^n\cdot f=\sum_{i=0}^{n}\binom{n}{i}(\Phi^{n-i}\cdot 1)(\theta^i\cdot f)$$

or

(3.14) $$\Phi^n\cdot f=\sum_{i=0}^{n}\binom{n}{i}(n-1)!\,(ax+b)^{-(n-i)\nu}\cdot(cx+d)^{-(n-i)\varepsilon}\cdot$$
$$\cdot S_{n-i}^{(\alpha-(n-i)\nu,\ \beta-(n-i)\varepsilon,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]\,\{\theta^i f\}$$

which will yield (3.9) when $f=1$.

Now,

$$\theta^m S_n^{(\alpha,\ \beta,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]=\frac{1}{n!}\sum_{r=0}^{m}\binom{m}{r}(m,-r)!\,(n+r)!\cdot$$
$$\cdot(ax+b)^{-\nu r}\cdot(cx+d)^{-\varepsilon r}\cdot S_{m-r}^{(-\alpha,\ -\beta,\ k)}\left[x,\ a,\ b,\ c,\ d;\ 0,\ 0;\ \frac{1}{\omega(x)}\right]\cdot$$
$$\cdot S_{n+r}^{(\alpha-\nu r,\ \beta-\varepsilon r,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]$$

which with the help of (3.4) gives

(3.15) $$\theta^m S_n^{(\alpha,\ \beta,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]=\sum_{r=0}^{m}\binom{m}{r}(m-r)!\cdot$$
$$\cdot S_{m-r}^{(-\alpha,\ -\beta,\ k)}\left[x,\ a,\ b,\ c,\ d;\ 0,\ 0;\ \frac{1}{\omega(x)}\right]\cdot$$
$$\cdot\Phi^r S_n^{(\alpha,\ \beta,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]$$

this suggests us,

(3.16) $$\theta^m=\sum_{r=0}^{m}\binom{m}{r}(m-r)!\,S_{m-r}^{(-\alpha,\ -\beta,\ k)}\left[x,\ a,\ b,\ c,\ d;\ \frac{1}{\omega(x)}\right]\Phi^r.$$

this relation is inverse to (3.14)

It can be easily verified that

$$e^{t\Phi}f(x)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\Phi^n\cdot 1\, e^{t\theta}\cdot f$$

hence,

(3.17) $$e^{t\Phi}f(x)=\sum_{n=0}^{\infty}t^n(ax+b)^{-n\nu}\cdot(cx+d)^{-n\varepsilon}\cdot$$
$$\cdot S_n^{(\alpha-n\nu,\ \beta-n\varepsilon,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]\cdot e^{t\theta}\cdot f$$

Now,

$$\sum_{n=0}^{\infty} t^n \cdot (ax+b)^{-n\nu} \cdot (cx+d)^{-n\varepsilon} S_n^{(\alpha-n\nu,\ \beta-n\varepsilon\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)] =$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{\omega(x)} \cdot \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot \theta^n \{(ax+b)^{\alpha} \cdot (cx+d)^{\beta} \cdot \omega(x)\}$$

which with the help of (1.17) yields

$$= \frac{(ax+b)^{-\alpha} \cdot cx+d)^{-\beta}}{\omega(x)} \cdot \left[\left(\frac{ax}{\{1-(k-1)\,tx^{k-1}\}^{1/k-1}} + b\right)^{\alpha} \cdot \left(\frac{cx}{\{1-(k-1)\,tx^{k-1}\}^{1/k-1}} + d\right)^{\beta} \cdot W\left(\frac{x}{\{1-(k-1)\,tx^{k-1}\}^{1/k-1}}\right) \cdot \right].$$

Hence, with the help of (1.17), we have

(3.18)
$$e^{t\Phi} f(x) = \sum_{n=0}^{\infty} t^n (ax+b)^{-n\nu} \cdot (cx+d)^{-n\varepsilon} \cdot S_n^{(\alpha-n\nu,\ \beta-n\varepsilon,\ k)} \cdot$$

$$\cdot [x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)] \cdot e^{t\theta} \cdot f$$

$$= \frac{(ax+b)^{-\alpha} \cdot (cx+d)^{-\beta}}{\omega(x)} \cdot \left[\left(\frac{ax}{\{1-(k-1)\,tx^{k-1}\}^{1/k-1}} + b\right)^{\alpha} \cdot \right.$$

$$\cdot \left(\frac{cx}{\{1-(t-1)\,x^{k-1}\}^{1/k-1}} + d\right)^{\beta} \cdot W\left(\frac{x}{\{1-(t-1)\,x^{k-1}\}^{1/k-1}}\right) \cdot f \cdot$$

$$\cdot \left\{\frac{x}{[1-(k-1)\,tx^{k-1}]^{1/k-1}}\right\}.$$

The generating relation of Chatterjea [2] is also a particular case of (3.18).

(3.19)
$$\sum_{n-0}^{\infty} F_n^{(r)}(x;\ a-kn,\ k,\ p)\ \frac{t^n}{n!} = (1+tx^{k-1})^a \cdot e^{pxr} \{1-(1+tx^{k-1})^r\}.$$

When $f(x) = S_n^{(\alpha,\ \beta,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]$ we have.

(3.20)
$$e^{t\Phi} S_n^{(\alpha,\ \beta,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)] =$$

$$= \frac{(ax+b)^{-\alpha} \cdot (cx+d)^{-\beta}}{\omega(x)} \cdot \left[\left(\frac{ax}{\{1-(k-1)\,tx^{k-1}\}^{1/k-1}} + b\right)^{\alpha} \cdot \right.$$

$$\cdot \left(\frac{cx}{\{1-(t-1)\,x^{k-1}\}^{1/k-1}} + d\right)^{\beta} \cdot W\left(\frac{x}{\{1-(t-1)\,x^{k-1}\}^{1/k-1}}\right) \cdot$$

$$\left. \cdot S_n^{(\alpha,\ \beta,\ k)}\left[\frac{x}{\{1-(t-1)\,x^{k-1}\}^{1/k-1}},\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)\right]\right].$$

and when $f(x)=1$, we have,

(3.21)
$$e^{t\Phi}\cdot 1=\sum_{n=0}^{\infty} t^n\cdot(ax+b)^{-n\nu}\cdot(cx+d)^{-n\varepsilon}\cdot$$
$$\cdot S_n^{(\alpha-n\nu,\ \beta-n\varepsilon,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]$$
$$=\frac{(ax+b)^{-\alpha}\cdot(cx+d)^{-\beta}}{\omega(x)}\cdot\left[\left(\frac{ax}{\{1-(t-1)x^{k-1}\}^{1/k-1}}+b\right)^{\alpha}\cdot\right.$$
$$\left.\cdot\left(\frac{cx}{\{1-(t-1)x^{k-1}\}^{1/k-1}}+d\right)^{\beta}\cdot W\left(\frac{x}{\{1-(t-1)x^{k-1}\}^{1/k-1}}\right)\right].$$

4. Linear generating relation

From (1.6) we have

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\ \beta,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]\,t^n=$$
$$=\sum_{n=1}^{\infty}\frac{(ax+b)^{-\alpha}\cdot(cx+d)^{-\beta}}{n!\,\omega(x)}(x^k D)^n\{(ax+b)^{\nu n+\alpha}\cdot(cx+d)^{\varepsilon n+\beta}\cdot\omega(x)\}.$$

Put $u=\dfrac{x^{-k+1}}{k+1}$, then $\dfrac{d}{du}=x^k\,\dfrac{d}{dx}$.

$$x=\{(1-k)\,u\}^{1/k-1}$$

Therefore we get,

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\ \beta,\ k)}[x,\ a,\ b,\ c,\ d;\ \nu,\ \varepsilon;\ \omega(x)]\,t^n=$$
$$=\frac{(ax+b)^{-\alpha}\cdot(cx+d)^{-\beta}}{\omega(x)}\cdot\sum_{n=0}^{\infty}\frac{t^n}{n!}\cdot\left(\frac{d}{du}\right)^n\cdot$$
$$\cdot\{[a((1-k)\,u)^{1/k-1}+b]^{\nu n+\alpha}\cdot[c((1-k)\,u)^{1/1-k}+d]^{\varepsilon n+\beta}\cdot$$
$$\cdot W(((1-k)\,u)^{1/1-k})\}.$$

Now we apply Lagrange's expansion [8] to simplify this result

(4.1)
$$\frac{f(z)}{1+t\,\Phi'(z)}=\sum_{n=0}^{\infty}\frac{t^n}{n!}\,D_x^n\{[\Phi(x)]^n\cdot f(x)\}$$

Where $Z=x+t\,\Phi(z)$

Let $\Phi(u)=(a((1-k)\,u)^{1/1-k}+b)^{\nu}\cdot(c((1-k)\,u)^{1/1-k}+d)^{\varepsilon}$

and $f(u)=((a((1-k)\,u)^{1/1-k}+b)^{\alpha}\cdot((c((1-k)\,u)^{1/1-k}+d)^{\beta}\cdot W(((1-k)\,u)^{1/1-k})$.

So we have

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\,\beta,\,k)}[x,\ a,\ b,\ c,\ d;_{\epsilon}\nu,\ \varepsilon;\ \omega(x)]\,t^n = \tag{4.2}$$

$$= \frac{(ax+b)^{-\alpha}\cdot(cx+d)^{-\beta}}{\omega(x)}\cdot$$

$$\cdot\frac{(a((1-k)z)^{1/1-k}+b)^{\alpha}\cdot(c((1-k)z)^{1/1-k}+d)^{\beta}\cdot W(((1-k)z)^{1/1-k})}{1-t\{\nu(a((1-k)z)^{1/1-k}+b)^{\nu-1}\cdot a((1-k)z)^{\frac{k}{1-k}}\cdot(c((1-k)z)^{1/1-k}+d)^{\varepsilon}+ +\varepsilon(c((1-k)z)^{1/1-k}+d)^{\varepsilon-1}\cdot c((1-k)z)^{\frac{k}{1-k}}(a((1-k)z)^{1/1-k}+b)^{\nu}\}}$$

where $Z=\dfrac{x^{-k+1}}{-k+1}+t(a((1-k)z)^{1/1-k}+b)^{\nu}\cdot(c((1-k)z)^{1/1-k}+d)^{\varepsilon}$.

This is the required generating relation.

Particular cases: — This generating function provides generalisation to the generating function of Shrivastava [10, *Eq.* 4.4).

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}F_n^{(r,\,m)}(x,\ a,\ k,\ p)=\left\{\frac{((1-k)z)^{1/1-k}}{x}\right\}^{a}\cdot \tag{4.3}$$

$$\cdot\{1-mt((1-k)z)^{\frac{m+k-1}{k-1}}\}^{-1}\cdot\exp\cdot[p\{x^r-((1-k)z)^{\frac{r}{1-k}}].$$

$$\text{where}\quad Z=\frac{x^{-k+1}}{-k+1}+t((1-k)z)^{\frac{m}{1-k}}.$$

Other particular cases of (4.2) are

$$\sum_{n=0}^{\infty}\frac{H_n(x)\,t^n}{n!}=e^{2xt-t^2}. \tag{4.4}$$

The generating relation of generalized Gould — Hopper [7]

$$\sum_{n=0}^{\infty}H_n^{(r)}(x,\ a,\ p)\,\frac{t^n}{n!}=x^{-a}(x-t)^{a}\cdot e^{p\{x^r-(x-t)^r\}}. \tag{4.5}$$

and the generating function for $F_n^{(r)}(x;\ a,\ k,\ p)$ given by Chatterjea [3]

$$\sum_{n=0}^{\infty}F_n^{(r)}(x;\ a,\ k,\ p)\,\frac{\omega^n}{n!}=\left(\frac{z}{x}\right)^{a}\cdot(1-\omega kz^{k-1})^{-1}\cdot\exp(p(x^2-r^2)) \tag{4.6}$$

$$\text{where}\quad Z=Wz^k+x.$$

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